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HYERS-ULAM STABILITY OF FOURTH ORDER EULER'S DIFFERENTIAL EQUATIONS

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Abstract. In this work, we investigate the Hyers-Ulam stability of the fourth order Euler's differential equations of the form

$$t^4y^{(iv)} + \alpha t^3 y^{'''} + \beta t^2 y^{''} + \gamma t y^{'} + \delta y = 0$$

on any open interval $I = (a, b)$, $0 < a < b \leq \infty$ or $-\infty < a < b < 0$, where α, β, γ and δ are complex constants.

1. Introduction

The study of stability problems for various functional equations originated from a talk given by S. M. Ulam in 1940. In that talk, Ulam [16] discussed a number of important unsolved problems. Among such problems, a problem concerning the stability of functional equations : *"Give conditions in order for a linear mapping near an approximately linear mapping to exist"* is one of them. In 1941, Hyers [1] gave an answer to the problem.

Furthermore, the result of Hyers [1] has been generalized by Rassias [12]. After that many authors have extended the Ulam's stability problems to other functional equations and generalized Hyer's result in various directions (see for e.g. [2, 6, 9, 10, 11, 13, 14]). Thereafter, Ulam's stability problem for functional equations was replaced by stability of differential equations. The differential equation

$$\begin{aligned} a_n(t)y^{(n)}(t) &+ a_{n-1}(t)y^{(n-1)}(t) + \cdots \\ &+ a_1(t)y^{'}(t) + a_0(t)y(t) + h(t) = 0 \end{aligned}$$

has Hyers-Ulam stability, if for given $\epsilon > 0$, I be an open interval and for any function f satisfying the differential inequality

$$\begin{aligned} |a_n(t)y^{(n)}(t) &+ a_{n-1}(t)y^{(n-1)}(t) + \cdots + \\ &+ a_1(t)y^{'}(t) + a_0(t)y(t) + h(t)| \leq \epsilon\,, \end{aligned}$$

then there exists a solution f_0 of the above differential equation such that $|f(t) - f_0(t)| \leq K(\epsilon)$ and $\lim_{\epsilon\to 0} K(\epsilon) = 0$, for $t \in I$. If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\psi(t)$ respectively, where ϕ, ψ: $I \to [0, \infty)$ are functions not depending on f and f_0 explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability.

S. M. Jung has investigated the Hyers-Ulam stability of linear differential equations of different classes. Among his works (see for e.g [3, 4, 5, 6, 7, 8]), we are motivated by the results of [4], where he has studied the Hyers-Ulam stability of the following Euler's differential equations:

$$ty^{'}(t) + \alpha y(t) + \beta t^r x_0 = 0$$

and also applied this result for the investigation of the Hyers-Ulam stability of the differential equation

$$t^2y^{''}(t) + \alpha t y^{'}(t) + \beta y(t) = 0,$$

where α, β and r are complex constants and $x_0 \neq 0$ is a fixed element. In [15], the authors have established the Hyers-Ulam stability of the following Euler's differential equations

$$t^2y^{''}(t) + \alpha t y^{'}(t) + \beta y(t) + \gamma t^r x_0 = 0 \tag{1.1}$$

and

$$t^3y^{'''}(t) + \alpha t^2 y^{''}(t) + \beta t y^{'}(t) + \gamma y(t) = 0, \tag{1.2}$$

where α, β, γ and r are complex constants with $x_0 \neq 0$ is a fixed element. In fact, Hyers-Ulam stability of (1.2) depends on the Hyers-Ulam stability of (1.1) for every $x_0 \neq 0$. In particular, we have the following results:

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Theorem 1.1. [15] *Let X be a Complex Banach space and let $I=(a,b)$ be an open interval with either $0<a<b\le\infty$ or $-\infty<a<b<0$. Assume that $\phi: I\to[0,\infty)$ is given, and that α,β,γ,r are complex constants, x_0 is a fixed element of X. Furthermore, suppose a twice continuously differentiable function $f: I\to X$ satisfies*

$$\left\|t^2y^{''}(t)+\alpha ty^{'}(t)+\beta y(t)+\gamma t^r x_0\right\|\le\phi(t),\ t\in I.$$

Let l and m be two numbers so that $\alpha=1-l-m$ and $\beta=lm$. If t^{r-l-1}, t^{r-m-1}, t^{m-l-1}, $t^{-m-1}\phi(t)$ and $t^{-l-1}\psi(t)$ are integrable on (a,c), for any c with $a<c\le b$, then there exists a unique solution $f_0: I\to X$ of the differential equation (1.1) such that

$$\|f(t)-f_0(t)\|\le\left|t^l\right|\left|\int_t^b u^{-l-1}\psi(u)du\right|$$

for all $t\in I$, where

$$\psi(t)=|t^m|\left|\int_t^b v^{-m-1}\phi(v)dv\right|.$$

Theorem 1.2. [15] *Let X be a Complex Banach space and $I=(a,b)$ be an open interval such that either $0<a<b\le\infty$ or $-\infty<a<b<0$. Assume that $\theta: I\to[0,\infty)$ is given along with α, β, and γ are complex constants and l, m and n are characteristic roots of (1.2), so that $\alpha=3-(l+m+n)$, $\beta=lm+mn+nl-l-m-n+1$ and $\gamma=-(lmn)$. Let $t^{n-l-1}, t^{n-m-1}, t^{m-l-1}, t^{-n-1}\theta(t), t^{-m-1}\lambda(t)$ and $t^{-l-1}\eta(t)$ are integrable on (a,c) with $a<c\le b$, where*

$$\lambda(t)=\left|t^n\int_t^b u^{-n-1}\theta(u)du\right|,$$

$$\eta(t)=\left|t^m\int_t^b v^{-m-1}\lambda(v)dv\right|.$$

Suppose that $f\in C^3(I,X)$ and satisfies

$$\left\|t^3y^{'''}(t)+\alpha t^2y^{''}(t)+\beta ty^{'}(t)+\gamma y(t)\right\|\le\theta(t),$$

for all $t\in I$. Then there exists a unique solution $f_0\in C^3(I,X)$ of (1.2) such that

$$\|f(t)-f_0(t)\|\le\left|t^l\int_t^b s^{-l-1}\eta(s)ds\right|.$$

Theorem 1.3. [15] *Let X be a Complex Banach space and let $I=(a,b)$ be an open interval such that $0<a<b\le\infty$ or $-\infty<a<b<0$. Assume that a function $\phi: I\to[0,\infty)$ is given and $h: I\to X$ is a continuous function. Furthermore, suppose a continuously differentiable function $f: I\to X$ satisfies*

$$\left\|ty^{'}(t)+\alpha y(t)+h(t)\right\|\le\phi(t),\ t\in I.$$

If both $t^{\alpha-1}\phi(t)$ and $t^{\alpha-1}h(t)$ are integrable on (a,c), for any c with $a<c\le b$, then there exists a unique solution $f_0: I\to X$ of the differential equation

$$ty^{'}(t)+\alpha y(t)+h(t)=0$$

such that

$$\|f(t)-f_0(t)\|\le\left|t^{-\alpha}\right|\left|\int_t^b v^{\alpha-1}\phi(v)dv\right|, t\in I,$$

where

$$f_0(t)=\left(\frac{a}{t}\right)^\alpha x-t^{-\alpha}\int_a^t u^{\alpha-1}h(u)du,$$

for unique $x\in X$ and α is a complex constant.

The aim of this work is to investigate the generalized Hyers-Ulam stability of the following Euler's differential equations of the form

$$t^3y^{'''}(t)+\alpha t^2y^{''}(t)+\beta ty^{'}(t)+\gamma y(t)+\delta t^r x_0=0 \tag{1.3}$$

and

$$t^4y^{(iv)}(t)+\alpha t^3y^{'''}(t)+\beta t^2y^{''}(t)+\gamma ty(t)+\delta y(t)=0, \tag{1.4}$$

where $\alpha,\beta,\gamma,\delta$ and r are complex constants with $x_0(\ne 0)\in X$. Here, we prove that if a function $f\in C^3(I,X)$ satisfies the differential inequality

$$\left\|t^3y^{'''}(t)+\alpha t^2y^{''}(t)+\beta ty^{'}(t)+\gamma y(t)+\delta t^r x_0\right\|\le\phi(t) \tag{1.5}$$

for all $t\in I$, where $x_0\in X$ be a fixed element and $I=(a,b)$ with $0<a<b\le\infty$ or $-\infty<a<b<0$, then there exists a unique solution $f_0\in C^3(I,X)$ of (1.5) such that

$$\|f(t)-f_0(t)\|\le|t^n|\left|\int_t^b u^{-n-1}\theta(u)du\right| \tag{1.6}$$

for all $t\in I$, where

$$\theta(t) = |t^l|\left|\int_t^b u^{-l-1}\psi(u)du\right|, \tag{1.7}$$

$$\psi(t) = |t^m|\left|\int_t^b v^{-m-1}\phi(v)dv\right|$$

and l,m,n are the characteristic roots of (1.3) such that $\alpha=3-(l+m+n)$, $\beta=lm+mn+nl-l-m-n+1$ and $\gamma=-(lmn)$. Also, we apply this result to investigate the Hyers-Ulam stability of (1.4).

Throughout this work, we let $I=(a,b)$ either $0<a<b\le\infty$ or $-\infty<a<b<0$.

2. Hyers-Ulam Stability

Let l, m and n be the characteristic roots of (1.3) such that $\alpha = 3-(l+m+n)$, $\beta = lm + mn + nl - l - m - n + 1$ and $\gamma = -(lmn)$ with $g(r) = r^3 + (\alpha-3)r^2 + (\beta-\alpha+2)r + \gamma$. For fixed $x_0 \neq 0$, the possible solutions of (1.3) in the class of real valued functions defined on I are given by

(I) when $g(r) \neq 0$,

$$y(t) = \begin{cases} c_1t^l + c_2t^m + c_3t^n - \frac{\delta x_0 t^r}{g(r)}, \\ for \quad l \neq m \neq n, \\ c_1t^l + (c_2 + c_3 ln\,|t|)t^m - \frac{\delta x_0 t^r}{g(r)}, \\ for \quad l \neq m = n, \\ (c_1 + c_2 ln\,|t|)\,t^l + c_3t^n - \frac{\delta x_0 t^r}{g(r)}, \\ for \quad l = m \neq n, \\ (c_1 + c_3 ln\,|t|)\,t^n + c_2t^m - \frac{\delta x_0 t^r}{g(r)}, \\ for \quad l = n \neq m, \\ \left\{c_1 + c_2 ln\,|t| + c_3\,(ln\,|t|)^2\right\} t^l - \frac{\delta x_0 t^r}{g(r)}, \\ for \quad l = m = n; \end{cases}$$

(II) when $g(r) = 0 \neq g'(r)$,

$$y(t) = \begin{cases} c_1t^l + c_2t^m + c_3t^n - \frac{\delta x_0 t^r ln|t|}{g'(r)}, \\ for \quad l \neq m \neq n, \\ c_1t^l + (c_2 + c_3 ln\,|t|)t^m - \frac{\delta x_0 t^r ln|t|}{g'(r)}, \\ for \quad r - l = 0, l \neq m = n, \\ (c_1 + c_2 ln\,|t|)\,t^l + c_3t^n - \frac{\delta x_0 t^r ln|t|}{g'(r)}, \\ for \quad r - n = 0, l = m \neq n, \\ (c_1 + c_3 ln\,|t|)\,t^n + c_2t^m - \frac{\delta x_0 t^r ln|t|}{g'(r)}, \\ for \quad r - m = 0, l = n \neq m. \end{cases}$$

Remark 2.1. *Indeed,* $g'(r) = 3r^2 + 2(\alpha-3)r + (\beta-\alpha+2)$*. If* $g(r) = 0$*, then either* $r - l = 0$ *or* $r - m = 0$ *or* $r - n = 0$*. Therefore,* $r - l = 0$ *and* $g'(r) \neq 0$ *implies that* $3r^2 + 2(\alpha-3)r + (\beta-\alpha+2) = (r-m)(r-n) \neq 0$*. So* $r - m \neq 0$ *and* $r - n \neq 0$*. Similarly, when* $r - m = 0$ *and* $g'(r) \neq 0$ *implies* $r - l \neq 0$ *and* $r - n \neq 0$ *and when* $r - n = 0$ *and* $g'(r) \neq 0$ *implies* $r - l =\neq 0$ *and* $r - m \neq 0$*. Hence the first solution for* $l \neq m \neq n$ *could be any one of the following:*

$$c_1t^l + c_2t^m + c_3t^n - \frac{\delta x_0 t^r ln\,|t|}{(r-m)(r-n)}$$
$$r - l = 0, \; r - m \neq 0, \; r - n \neq 0,$$

$$c_1t^l + c_2t^m + c_3t^n - \frac{\delta x_0 t^r ln\,|t|}{(r-l)(r-m)};$$
$$r - n = 0, \; r - l \neq 0, \; r - m \neq 0,$$

$$c_1t^l + c_2t^m + c_3t^n - \frac{\delta x_0 t^r ln\,|t|}{(r-l)(r-n)};$$
$$r - m = 0, \; r - l \neq 0, \; r - n \neq 0.$$

(III) When $g(r) = 0 = g'(r)$, but $g''(r) \neq 0$

$$y(t) = \begin{cases} (c_1 + c_2 ln\,|t|)\,t^l + c_3t^n - \frac{\delta x_0 t^r (ln|t|)^2}{g''(r)}, \\ for \quad l = m = r, \; r - n \neq 0, \\ c_1t^l + (c_2 + c_3 ln\,|t|)t^m - \frac{\delta x_0 t^r (ln|t|)^2}{g''(r)}, \\ for \quad m = n = r, \; r - l \neq 0, \\ (c_1 + c_3 ln\,|t|)\,t^n + c_2t^m - \frac{\delta x_0 t^r (ln|t|)^2}{g''(r)}, \\ for \quad l = n = r, \; r - m \neq 0. \end{cases}$$

(IV) When $g(r) = 0 = g'(r) = g''(r)$,

$$y(t) = \left\{c_1 + c_2 ln\,|t| + c_3\,(ln\,|t|)^2\right\} t^l - \frac{\delta x_0 t^r\,(ln\,|t|)^3}{g'''(r)},$$

where c_1, c_2 and c_3 are arbitrary constants.

Theorem 2.1. *Let* X *be a Complex Banach space. Assume that a function* $\phi : I \to [0, \infty)$ *is given, that* $\alpha, \beta, \gamma, \delta, r$ *are complex constants and that* x_0 *is a fixed element of* X *. Furthermore, suppose a thrice continuously differentiable function* $f : I \to X$ *satisfies the differential inequality (1.5). If* t^{r-l-1}, t^{m-l-1}, t^{l-n-1}, t^{m-n-1}, t^{r-n-1}, $t^{m-l-1} ln\left|\frac{t}{a}\right|$, $t^{l-n-1} ln\left|\frac{t}{a}\right|$, $t^{r-n-1} ln\left|\frac{t}{a}\right|$, $t^{r-n-1}\left(ln\left|\frac{t}{a}\right|\right)^2$, $t^{-m-1}\phi(t)$, $t^{-l-1}\psi(t)$ *and* $t^{-n-1}\theta(t)$ *are integrable on* (a, c)*, for any* c *with* $a < c \leq b$*, then there exists a unique solution* $f_0 \in C^3(I, X)$ *of (1.3) such that (1.6) holds for any* $t \in I$*, where* l, m *and* n *are the roots of* $g(r) = 0$.

Proof. To prove the theorem it is sufficient to consider the following cases:

(i) $(r-l)(r-m)(r-n) \neq 0, \; l \neq m \neq n;$
(ii) $(r-l)(r-m)(r-n) \neq 0, \; l \neq m = n;$
(iii) $(r-l)(r-m)(r-n) \neq 0, \; l = m \neq n;$
(iv) $(r-l)(r-m)(r-n) \neq 0, \; l = n \neq m;$
(v) $(r-l)(r-m)(r-n) \neq 0, \; l = m = n;$
(vi) $(r-l) = 0, (r-m) \neq 0 \neq (r-n), \; l \neq m \neq n;$
(vii) $(r-n) = 0, (r-m) \neq 0 \neq (r-l), \; l \neq m \neq n;$
(viii) $(r-m) = 0, (r-l) \neq 0 \neq (r-n), \; l \neq m \neq n;$
(ix) $(r-l) = 0, (r-m) \neq 0 \neq (r-n), \; l \neq m = n;$
(x) $(r-n) = 0, (r-m) \neq 0 \neq (r-l), \; l = m \neq n;$
(xi) $(r-m) = 0, (r-l) \neq 0 \neq (r-n), \; l = n \neq m;$
(xii) $(r-l) = 0 = (r-m), (r-n) \neq 0, \; l = m \neq n;$
(xiii) $(r-m) = 0 = (r-n), (r-l) \neq 0, \; l \neq m = n;$
(xiv) $(r-l) = 0 = (r-n), (r-m) \neq 0, \; l = n \neq m;$
(xv) $(r-l) = (r-m) = (r-n) = 0, \; l = m = n.$

Case-(i) Suppose that X is a Complex Banach space and a thrice continuously differentiable function $f : I \to X$ is satisfying the differential inequality (1.3). Let l, m and n be the roots of $g(u) = u^3 + (\alpha-3)u^2 + (\beta-\alpha+2)u + \gamma = 0$. Define $h : I \to X$

such that $h(t) = tf^{'}(t) - nf(t)$. Then we have

$$\left\| t^2\ h^{''}(t) + (1-l-m)t\ h^{'}(t) + lm\ h(t) + \delta\ t^r x_0 \right\| = \left\| t^3\ f^{'''}(t) + \alpha\ t^2\ f^{''}(t) + \beta\ t\ f^{'}(t) + \gamma\ f(t) + \delta\ t^r x_0 \right\| \leq\ \phi(t).$$

Hence by Theorem 1.1 and then using (1.7), it follows that there exists a unique solution $h_0 : I \to X$ of the differential equation

$$t^2 y^{''}(t) + (1-l-m)ty^{'}(t) + lmy(t) + \delta t^r x_0 = 0$$

such that

$$\|h(t) - h_0(t)\| \leq |t^l| \left| \int_t^b u^{-l-1}\psi(u)du \right| = \theta(t), \tag{2.1}$$

where $h_0(t) = k_1 t^l + k_2 t^m + k_3 t^r$ with

$$k_1 = \left\{ \frac{x_1}{a^l} - \frac{\delta x_0 a^{r-l}}{(r-l)(m-l)} - \frac{x}{a^l(m-l)} \right\},$$
$$k_2 = \left\{ \frac{x}{a^m(m-l)} - \frac{\delta x_0 a^{r-m}}{(r-m)(l-m)} \right\},$$
$$k_3 = -\frac{\delta x_0}{(r-l)(r-m)}.$$

Consequently, (2.1) becomes

$$\left\| tf^{'}(t) - nf(t) - h_0(t) \right\| \leq \theta(t).$$

Clearly $\theta : I \to [0, \infty)$. By Theorem 1.3, there exists a unique solution $f_0 : I \to X$ of the differential equation $ty^{'}(t) - ny(t) - h_0(t) = 0$ such that

$$\|f(t) - f_0(t)\| \leq |t^n| \left| \int_t^b v^{-n-1}\theta(v)dv \right|,\ t \in I$$

provided $t^{-n-1}\theta(t)$ and $t^{-n-1}h_0(t)$ are integrable on (a, c), for any c with $a < c \leq b$. As $t^{l-n-1}, t^{m-n-1}, t^{r-n-1}, t^{l-n-1}ln\left|\frac{t}{a}\right|, t^{r-n-1}ln\left|\frac{t}{a}\right|$ and $t^{l-n-1}\left(ln\left|\frac{t}{a}\right|\right)^2$ are integrable, then so also $t^{-n-1}h_0(t)$. According to Theorem 1.3, $f_0(t)$ is given by

$$f_0(t) = \left(\frac{t}{a}\right)^n \bar{x} + t^n \int_a^t v^{-n-1}h_0(v)dv, \tag{2.2}$$

where $\bar{x}$ is a limit point in X. It is easy to verify that

$$\int_a^t v^{-n-1}h_0(v)dv = \frac{k_1}{l-n}\left(t^{l-n} - a^{l-n}\right) + \frac{k_2}{m-n}\left(t^{m-n} - a^{m-n}\right) + \frac{k_3}{r-n}\left(t^{r-n} - a^{r-n}\right).$$

As a result,

$$f_0(t) = \frac{k_1}{l-n}t^l + \frac{k_2}{m-n}t^m + \left\{ \frac{\bar{x}}{a^n} - \frac{k_1 a^{l-n}}{l-n} - \frac{k_2 a^{m-n}}{m-n} - \frac{k_3 a^{r-n}}{r-n} \right\} t^n - \frac{\delta x_0 t^r}{g(r)}.$$

Case-(ii) Proceeding as in Case (i), we can obtain

$$\int_a^t v^{-n-1}h_0(v)dv = \frac{k_1}{l-n}\left(t^{l-n} - a^{l-n}\right) + k_2\ ln\left|\frac{t}{a}\right| + \frac{k_3}{r-n}\left(t^{r-n} - a^{r-n}\right).$$

Consequently,

$$f_0(t) = \frac{k_1}{l-m}t^l + \left\{ \frac{\bar{x}}{a^m} - \frac{k_1 a^{l-m}}{l-m} - \frac{k_3 a^{r-m}}{r-m} + k_2\ ln\left|\frac{t}{a}\right| \right\} t^m - \frac{\delta x_0 t^r}{(r-l)(r-m)^2},$$

where k_1, k_2 and k_3 are same as in Case(i).

Case-(iii) Proceeding as before with

$$h_0(t) = \{k_4 + k_5 ln\left|\frac{t}{a}\right|\}t^l + k_6 t^r$$

and

$$k_4 = \left\{ \frac{x_2}{a^l} + \frac{\delta x_0 a^{r-l}}{(r-l)^2} \right\},$$
$$k_5 = \left\{ \frac{x}{a^l} + \frac{\delta x_0 a^{r-l}}{r-l} \right\},$$
$$k_6 = -\frac{\delta x_0}{(r-l)^2},$$

it is easy to verify that

$$\int_a^t v^{-n-1}h_0(v)dv = \frac{k_4}{l-n}\left(t^{l-n} - a^{l-n}\right) + \frac{k_5}{l-n}\left[\left(ln\left|\frac{t}{a}\right| - \frac{1}{l-n}\right)\ t^{l-n} + \frac{a^{l-n}}{(l-n)^2}\right] + \frac{k_6}{r-n}\left(t^{r-n} - a^{r-n}\right).$$

Hence, from (2.2)

$$f_0(t) = \left[\frac{k_4}{l-n} - \frac{k_5}{(l-n)^2} + \frac{k_{11}}{l-n}ln\left|\frac{t}{a}\right|\right] t^l + \left[\frac{\bar{x}}{a^n} - \frac{k_4 a^{l-n}}{l-n} + \frac{k_5 a^{l-n}}{(l-n)^2} - \frac{k_6 a^{r-n}}{r-n}\right] t^n - \frac{\delta x_0 t^r}{(r-n)(r-l)^2}.$$

Case-(iv) We proceed as in Case (i) and it is easy to see that

$$\int_a^t v^{-n-1}h_0(v)dv = k_1 ln\left|\frac{t}{a}\right| + \frac{k_2}{m-n}\left(t^{m-n} - a^{m-n}\right) + \frac{k_3}{r-n}\left(t^{r-n} - a^{r-n}\right).$$

So from (2.2), it happens that

$$f_0(t) = \frac{k_2}{m-n}t^m + \left\{ \frac{\bar{x}}{a^n} - \frac{k_2 a^{m-n}}{m-n} - \frac{k_3 a^{r-n}}{r-n} + k_1 ln\left|\frac{t}{a}\right| \right\} t^n - \frac{\delta x_0 t^r}{(r-m)(r-n)^2}.$$

Case-(v) In this case we have that

$$h_0(t) = \left\{k_4 + k_5 ln\left|\frac{t}{a}\right|\right\} t^l + k_6 t^r,$$

where k_4, k_5 and k_6 are same as in Case (iii). Therefore,

$$\int_a^t v^{-n-1} h_0(v)dv = k_4\ ln\left|\frac{t}{a}\right| + \frac{k_5}{2}\left(ln\left|\frac{t}{a}\right|\right)^2 + \frac{k_6}{r-n}\left(t^{r-n} - a^{r-n}\right)$$

implies from (2.2) that

$$f_0(t) = \left[\frac{\bar{x}}{a^n} - \frac{k_6 a^{r-l}}{r-l} + k_4 ln\left|\frac{t}{a}\right| + \frac{k_5}{2}\left(ln\left|\frac{t}{a}\right|\right)^2\right] t^l - \frac{\delta x_0 t^r}{(r-l)^3}.$$

Case-(vi) In this case, we notice that

$$h_0(t) = k_7 t^l + k_8 t^m + k_9 t^r ln\left|\frac{t}{a}\right|,$$

where

$$k_7 = \left\{\frac{x_3}{a^l} + \frac{\delta x_0}{(r-m)^2} + \frac{x}{a^l(r-m)}\right\},$$
$$k_8 = \left\{\frac{x}{a^m(m-l)} - \frac{\delta x_0 a^{r-m}}{(r-m)^2}\right\},$$
$$k_9 = -\frac{\delta x_0}{(r-m)},$$

and

$$\int_a^t v^{-n-1} h_0(v)dv = \frac{k_7}{l-n}\left(t^{l-n} - a^{l-n}\right) + \frac{k_8}{m-n}\left(t^{m-n} - a^{m-n}\right) + k_9\left[\frac{t^r}{r-n} ln\left|\frac{t}{a}\right| - \frac{t^{r-n}}{(r-n)^2} - \frac{a^{r-n}}{(r-n)^2}\right].$$

Hence from (2.2), it follows that

$$f_0(t) = \left[\frac{k_7}{l-n} - \frac{k_9}{(l-n)^2}\right] t^l + \frac{k_8}{m-n} t^m + \left[\frac{\bar{x}}{a^n} - \frac{k_7\ a^{l-n}}{l-n} - \frac{k_8\ a^{m-n}}{m-n} + \frac{k_9\ a^{l-n}}{(l-n)^2}\right] t^n - \frac{\delta x_0 t^r}{(r-m)(r-n)}\ ln\left|\frac{t}{a}\right|.$$

Case-(vii) Here, we have

$$h_0(t) = k_1 t^l + k_2 t^m + k_3 t^r,$$

and

$$\int_a^t v^{-n-1} h_0(v)dv = \frac{k_1}{l-n}\left(t^{l-n} - a^{l-n}\right) + \frac{k_2}{m-n}\left(t^{m-n} - a^{m-n}\right) + k_3 ln\left|\frac{t}{a}\right|,$$

where k_1, k_2 and k_3 are same as in Case-(i). Applying this in (2.2), we obtain

$$f_0(t) = \frac{k_1}{l-n} t^l + \frac{k_2}{m-n} t^m + \left[\frac{\bar{x}}{a^n} - \frac{k_1 a^{l-n}}{l-n} - \frac{k_2 a^{m-n}}{m-n}\right] t^n - \frac{\delta x_0 t^r}{(r-l)(r-m)}\ ln\left|\frac{t}{a}\right|.$$

Case-(viii) For this case, $h_0(t)$ becomes

$$h_0(t) = k_{10} t^l + k_{11} t^m + k_{12} t^r ln\left|\frac{t}{a}\right|,$$

where

$$k_{10} = \left\{\frac{x_4}{a^l} - \frac{\delta x_0 a^{m-l}}{(m-l)^2} - \frac{x}{a^l(m-l)}\right\},$$
$$k_{11} = \left\{\frac{x}{a^m(m-l)} + \frac{\delta x_0}{(m-l)^2}\right\},$$
$$k_{12} = -\frac{\delta x_0}{(r-l)}.$$

Using

$$\int_a^t v^{-n-1} h_0(v)dv = \frac{k_{10}}{l-n}\left(t^{l-n} - a^{l-n}\right) + \frac{k_{11}}{m-n}\left(t^{m-n} - a^{m-n}\right) + \frac{k_{12}}{r-n}\left[t^{r-n} ln\left|\frac{t}{a}\right| - \frac{t^{r-n}}{r-n} + \frac{a^{r-n}}{(r-n)^2}\right]$$

in (2.2), we find

$$f_0(t) = \left(\frac{k_{10}}{l-n}\right) t^l + \left\{\frac{k_{11}}{m-n} + \frac{\delta x_0}{(m-l)(m-n)^2}\right\} t^m + \left[\frac{\bar{x}}{a^n} - \frac{k_{10}\ a^{l-n}}{l-n} - \frac{k_{11}\ a^{m-n}}{m-n} - \frac{\delta x_0\ a^{m-n}}{(m-l)(m-n)^2}\right] t^n - \frac{\delta x_0 t^r}{(r-l)(r-n)}\ ln\left|\frac{t}{a}\right|.$$

Case-(ix) Here $h_0(t) = k_7 t^l + k_8 t^m + k_9 t^r ln\left|\frac{t}{a}\right|$, where k_7, k_8 and k_9 are same as in Case-(vi). Using the fact

$$\int_a^t v^{-n-1} h_0(v)dv = \frac{k_7}{l-n}\left(t^{l-n} - a^{l-n}\right) + k_8\ ln\left|\frac{t}{a}\right| + \frac{k_9}{r-n}\left[t^{r-n} ln\left|\frac{t}{a}\right| - \frac{t^{r-n}}{r-n} + \frac{a^{r-n}}{r-n}\right]$$

in (2.2), it follows that

$$f_0(t) = \left(\frac{k_7}{l-m} - \frac{k_9}{(l-m)^2}\right) t^l + \left[\frac{\bar{x}}{a^m} - \frac{k_7\ a^{l-m}}{l-m} + \frac{k_9\ a^{l-m}}{(l-m)^2} + k_8 ln\left|\frac{t}{a}\right|\right] t^m - \frac{\delta x_0 t^r}{(r-m)^2}\ ln\left|\frac{t}{a}\right|.$$

Case-(x) In this case, $h_0(t) = \left\{k_4 + k_5 ln\left|\frac{t}{a}\right|\right\} t^l + k_6 t^r$, where k_4, k_5 and k_6 are same as in Case-(iii).

Now, it is easy to verify that

$$\int_a^t v^{-n-1}h_0(v)dv = \frac{k_4}{l-n}\left(t^{l-n}-a^{l-n}\right)+ + \frac{k_5}{l-n}\left[t^{l-n}\ ln\left|\frac{t}{a}\right| - \frac{t^{l-n}}{l-n}+\frac{a^{l-n}}{l-n}\right]+k_6\ ln\left|\frac{t}{a}\right|.$$

Hence, from (2.2) we obtain

$$f_0(t) = \left[\left\{\frac{k_4}{l-n}-\frac{k_5}{(l-n)^2}\right\}+\frac{k_5}{l-n}\ ln\left|\frac{t}{a}\right|\right]t^l + \left[\frac{\bar{x}}{a^n}-\frac{k_4\ a^{l-n}}{l-n}+\frac{k_5\ a^{l-n}}{(l-n)^2}\right]t^n - \frac{\delta x_0 t^r}{(r-l)^2}\ ln\left|\frac{t}{a}\right|.$$

Case-(xi) In this case, $h_0(t) = k_{10}t^l + k_{11}t^m + k_{12}t^r ln\left|\frac{t}{a}\right|$, where k_{10}, k_{11} and k_{12} are same as in Case-($viii$). Clearly,

$$\int_a^t v^{-n-1}h_0(v)dv = k_{10}\ ln\left|\frac{t}{a}\right| + \frac{k_{11}}{m-n}\left(t^{m-n}-a^{m-n}\right)+ \frac{k_{12}}{r-n}\left[t^{r-n}\ ln\left|\frac{t}{a}\right|-\frac{t^{r-n}}{r-n}+\frac{a^{r-n}}{r-n}\right]$$

and (2.2) reduces to

$$f_0(t) = \left[\frac{\bar{x}}{a^l}-\frac{k_{11}\ a^{m-l}}{m-l}+\frac{k_{12}\ a^{m-l}}{(m-l)^2}+k_{10}\ ln\left|\frac{t}{a}\right|\right]t^l + \left[\frac{k_{11}}{m-l}-\frac{k_{12}}{(m-l)^2}\right]t^m - \frac{\delta x_0 t^r}{(r-n)^2}\ ln\left|\frac{t}{a}\right|.$$

Case-(xii) In this case we obtain

$$h_0(t) = \left\{\frac{x_5}{a^l}+\frac{x}{a^l}\ ln\left|\frac{t}{a}\right|\right\}t^l - \frac{\delta x_0 t^r}{2}\left(ln\left|\frac{t}{a}\right|\right)^2$$

and thus

$$\int_a^t v^{-n-1}h_0(v)dv = \frac{x_5}{a^l(l-n)}\left[t^{l-n}-a^{l-n}\right]+ \frac{x}{a^l(l-n)}\left[t^{l-n}\ ln\left|\frac{t}{a}\right|-\frac{t^{l-n}}{l-n}+\frac{a^{l-n}}{l-n}\right]-\frac{\delta x_0}{2(r-n)}\times \times\left[t^{r-n}\left(ln\left|\frac{t}{a}\right|\right)^2-\frac{2t^{r-n}}{r-n}ln\left|\frac{t}{a}\right|+\frac{2t^{r-n}}{(r-n)^2}-\frac{2a^{r-n}}{(r-n)^2}\right]$$

which on applying in (2.2), we get

$$f_0(t) = \left[\frac{x_5}{a^l(l-n)}-\frac{x}{a^l(l-n)^2}-\frac{\delta x_0}{(l-n)^3}+ \left\{\frac{x}{a^l(l-n)}+\frac{\delta x_0}{(l-n)^2}\right\}ln\left|\frac{t}{a}\right|\right]t^l + \left[\frac{\bar{x}}{a^n}-\frac{x_5}{a^n(l-n)}+\frac{x}{a^n(l-n)^2}+\frac{\delta x_0 a^{l-n}}{(l-n)^3}\right]t^n - \frac{\delta x_0 t^r}{2(r-n)}\left(ln\left|\frac{t}{a}\right|\right)^2.$$

Case-(xiii) Here $h_0(t) = k_{10}t^l + k_{11}t^m + k_{12}t^r ln\left|\frac{t}{a}\right|$, where k_{10}, k_{11} and k_{12} are same as in Case-($viii$) and

$$\int_a^t v^{-n-1}h_0(v)dv = \frac{k_{10}}{l-n}\left(t^{l-n}-a^{l-n}\right)+ + k_{11}\ ln\left|\frac{t}{a}\right|+\frac{k_{12}}{2}\left(ln\left|\frac{t}{a}\right|\right)^2.$$

Applying this in (2.2), it follows that

$$f_0(t) = \frac{k_{10}}{l-n}t^l + \left\{\frac{\bar{x}}{a^n}-\frac{k_{10}\ a^{l-n}}{l-n}+k_{11}\ ln\left|\frac{t}{a}\right|\right\}t^n - \frac{\delta x_0 t^r}{2(r-l)}\left(ln\left|\frac{t}{a}\right|\right)^2.$$

Case-(xiv) In this case,

$$h_0(t) = k_7t^l + k_8t^m + k_9t^r ln\left|\frac{t}{a}\right|,$$

where k_7, k_8 and k_9 are same as in Case-(vi) and

$$\int_a^t v^{-n-1}h_0(v)dv = k_7\ ln\left|\frac{t}{a}\right| + \frac{k_8}{m-n}\left(t^{m-n}-a^{m-n}\right)+\frac{k_9}{2}\left(ln\left|\frac{t}{a}\right|\right)^2.$$

Ultimately, (2.2) becomes

$$f_0(t) = \left(\frac{k_8}{m-n}\right)t^m + \left\{\frac{\bar{x}}{a^n}-\frac{k_8\ a^{m-n}}{m-n}+k_7\ ln\left|\frac{t}{a}\right|\right\}t^n - \frac{\delta x_0 t^r}{2(r-m)}\left(ln\left|\frac{t}{a}\right|\right)^2.$$

Case-(xv) For this case,

$$h_0(t) = \left\{\frac{x_5}{a^l}+\frac{x}{a^l}\ ln\left|\frac{t}{a}\right|\right\}t^l - \frac{\delta x_0 t^r}{2}\left(ln\left|\frac{t}{a}\right|\right)^2$$

and hence

$$\int_a^t v^{-n-1}h_0(v)dv = \frac{x_5}{a^l}\ ln\left|\frac{t}{a}\right|+\frac{x}{2a^l}\left(ln\left|\frac{t}{a}\right|\right)^2 - \frac{\delta x_0}{6}\left(ln\left|\frac{t}{a}\right|\right)^3.$$

Using this in (2.2), $f_0(t)$ can be obtained as

$$f_0(t) = \left[\frac{\bar{x}}{a^l}+\frac{x_5}{a^l}\ ln\left|\frac{t}{a}\right|+\frac{x}{2a^l}\left(ln\left|\frac{t}{a}\right|\right)^2\right]t^l - \frac{\delta x_0 t^r}{6}\left(ln\left|\frac{t}{a}\right|\right)^3.$$

Here $\bar{x}, x, x_1, x_2, x_3, x_4$ and x_5 are all limit points in X. This completes the proof of the theorem. □

3. Main results

In this section, we investigate the Hyers-Ulam stability of (1.4) on I. Assume that l, m, n and p are the characteristic roots (1.4) such that

$$\begin{aligned} \alpha &= \{6-(l+m+n+p)\}, \\ \beta &= \{7-3(l+m+n+p)+ \\ &\quad +(lm+mn+nl+np+lp+mp)\}, \\ \gamma &= \{1-(l+m+n+p)+ \\ &\quad +(lm+mn+nl+mp+lp+np)- \\ &\quad -(lmn+mnp+lnp+lmp)\}, \\ \delta &= lmnp,\ \alpha_1=\{3-(l+m+n)\}, \\ \beta_1 &= \{lm+mn+nl-l-m-n+1\} \text{ and} \\ \gamma_1 &= -(lmn). \end{aligned}$$

Theorem 3.1. *Let X be a Complex Banach space. Assume that a function $\eta : I \to [0,\infty)$ is given. Furthermore assume that t^{p-l-1}, t^{m-l-1}, t^{l-n-1}, t^{m-n-1}, t^{p-n-1}, $t^{m-l-1}ln\left|\frac{t}{a}\right|$, $t^{l-n-1}ln\left|\frac{t}{a}\right|$, $t^{p-n-1}ln\left|\frac{t}{a}\right|$, $t^{p-n-1}\left(ln\left|\frac{t}{a}\right|\right)^2$, $t^{-p-1}\eta(t), t^{-m-1}\phi(t)$, $t^{-l-1}\psi(t)$ and $t^{-n-1}\theta(t)$ are integrable over the interval (a,c) with $a < c \le b$, where*

$$\begin{aligned} \phi(t) &= |t^p|\left|\int_t^b v^{-p-1}\eta(v)dv\right|, \\ \psi(t) &= |t^m|\left|\int_t^b v^{-m-1}\phi(v)dv\right|, \\ \theta(t) &= |t^l|\left|\int_t^b u^{-l-1}\psi(u)du\right|. \end{aligned}$$

Suppose that $f \in C^4(I,X)$ satisfies the differential inequality

$$\left\|t^4y^{(iv)}(t)+\alpha t^3y'''(t)+\beta t^2y''(t)+\gamma ty(t)+\delta y(t)\right\| \le \eta(t), \qquad (3.1)$$

for all $t \in I$. Then there exists a unique solution $f_0 \in C^4(I,X)$ of (1.4) such that

$$\|f(t)-f_0(t)\| \le \left|t^n\int_t^b v^{-n-1}\theta(v)dv\right|.$$

Proof. Let X be a Complex Banach space and $f : I \to X$ such that (3.1) hold, for $t \in I$. Define $s : I \to X$ such that

$$s(t) = t^3f'''(t)+\alpha_1t^2f''(t)+\beta_1tf'(t)+\gamma_1f(t). \qquad (3.2)$$

Indeed,

$$\|ts'(t)-ps(t)\| = \left\|t^4f^{(iv)}(t)+\alpha t^3f'''(t)+\beta t^2f''(t)+\gamma tf'(t)+\delta f(t)\right\| \le \eta(t).$$

From Theorem 1.3, it follows that there exists a unique solution $s_0 : I \to X$ of the differential equation $ts'(t)-ps(t)=0$ such that

$$\|s(t)-s_0(t)\| \le |t^p|\left|\int_t^b u^{-p-1}\eta(u)du\right|,$$

where $s_0(t)=\left(\frac{t}{a}\right)^px_0$ and $x_0 \in X$ is a limit point. If we denote

$$\phi(t) = |t^p|\left|\int_t^b u^{-p-1}\eta(u)du\right|,$$

then clearly $\phi : I \to [0,\infty)$ and

$$\|s(t)-s_0(t)\| \le \phi(t). \qquad (3.3)$$

Therefore, from (3.2) and (3.3) we get

$$\left\|t^3f'''(t)+\alpha_1t^2f''(t)+\beta_1tf'(t)+\gamma_1f(t)-a^{-p}t^px_0\right\| \le \phi(t) \qquad (3.4)$$

Using Theorem 2.2 in (3.4), it follows that there exists a unique solution $f_0 : I \to X$ such that

$$\|f(t)-f_0(t)\| \le |t^n|\left|\int_t^b u^{-n-1}\theta(u)du\right|,$$

where

$$\begin{aligned} \theta(t) &= \left|t^l\right|\left|\int_t^b u^{-l-1}\psi(u)du\right|, \\ \psi(t) &= |t^m|\left|\int_t^b u^{-m-1}\phi(u)du\right|, \\ \phi(t) &= |t^p|\left|\int_t^b u^{-p-1}\eta(u)du\right|. \end{aligned}$$

Here, $f_0(t)$ can be made any of the following cases

(i)
$$\begin{aligned} f_0(t) &= \frac{e_1}{l-n}t^l+\frac{e_2}{m-n}t^m+ \\ &\quad +\left\{\frac{\bar{x}}{a^n}-\frac{e_1a^{l-n}}{l-n}-\frac{e_2a^{m-n}}{m-n}-\frac{e_3a^{p-n}}{p-n}\right\}t^n+ \\ &\quad +\frac{x_0}{a^p(p-l)(p-m)(p-n)}\,t^p, \end{aligned}$$

where

$$\begin{aligned} e_1 &= \left\{\frac{x_1}{a^l}+\frac{x_0}{a^l(p-l)(m-l)}-\frac{x}{a^l(m-l)}\right\}, \\ e_2 &= \left\{\frac{x}{a^m(m-l)}+\frac{x_0}{a^m(p-m)(l-m)}\right\} \text{ and} \\ e_3 &= \frac{x_0}{a^p(p-l)(p-m)}. \end{aligned}$$

(ii)
$$\begin{aligned} f_0(t) &= \left(\frac{e_1}{l-n}\right)t^l+ \\ &\quad +\left\{\frac{\bar{x}}{a^n}-\frac{e_1a^{l-n}}{l-n}-\frac{e_3a^{p-n}}{p-n}+e_2ln\left|\frac{t}{a}\right|\right\}t^n+ \\ &\quad +\frac{x_0}{a^p(p-l)(p-m)^2}\,t^p. \end{aligned}$$

$$(iii)\quad f_0(t) = \left\{\frac{e_4}{l-n} - \frac{e_5}{(l-n)^2} + \frac{e_5}{l-n} ln\left|\frac{t}{a}\right|\right\} t^l + \left\{\frac{\bar{x}}{a^n} - \frac{e_4 a^{l-n}}{l-n} + \frac{e_5 a^{l-n}}{(l-n)^2} - \frac{e_6 a^{p-n}}{p-n}\right\} t^n + \frac{x_0}{a^p(p-n)(p-l)^2}\ t^p ,$$

where

$$e_4 = \left\{\frac{x_2}{a^l} - \frac{x_0}{a^l(p-l)}\right\},\quad e_5 = \left\{\frac{x}{a^l} - \frac{x_0}{a^l(p-l)}\right\}\ and\quad e_6 = \frac{x_0}{a^p(p-l)^2}.$$

$$(iv)\ f_0(t) = \left(\frac{e_2}{m-n}\right) t^m + \left\{\frac{\bar{x}}{a^n} - \frac{e_2\ a^{m-n}}{m-n} - \frac{e_3\ a^{p-n}}{p-n} + e_1 ln\left|\frac{t}{a}\right|\right\} t^n + \frac{x_0}{a^p(p-m)(p-n)^2}\ t^p .$$

$$(v)\ f_0(t) = \left\{\frac{\bar{x}}{a^n} - \frac{e_6\ a^{p-n}}{p-n} + e_4\ ln\left|\frac{t}{a}\right| + \frac{e_5}{2}\left(ln\left|\frac{t}{a}\right|\right)^2\right\} t^l + \frac{x_0}{a^p(p-l)^3}\ t^p .$$

$$(vi)\ f_0(t) = \frac{e_7}{l-n} t^l + \frac{e_8}{m-n} t^m + \left\{\frac{\bar{x}}{a^n} - \frac{e_7 a^{l-n}}{l-n} - \frac{e_8 a^{m-n}}{m-n} - \frac{e_9 a^{p-n}}{(p-n)^2}\right\} t^n + \left[\frac{e_9}{p-n} ln\left|\frac{t}{a}\right| - \frac{e_9}{(p-n)^2}\right]\ t^p ,$$

where

$$e_7 = \left\{\frac{x_3}{a^l} - \frac{x_0}{a^l(l-m)^2} + \frac{x}{a^l(l-m)}\right\},\quad e_8 = \left\{\frac{x}{a^m(m-l)} + \frac{x_0}{a^m(m-l)^2}\right\}\ and\quad e_9 = \frac{x_0}{a^p(p-m)}.$$

$$(vii)\ f_0(t) = \frac{e_1}{l-n} t^l + \frac{e_2}{m-n} t^m + \left\{\frac{\bar{x}}{a^n} - \frac{e_1 a^{l-n}}{l-n} - \frac{e_2 a^{m-n}}{m-n}\right\} t^n + \frac{x_0}{a^p(p-l)(p-m)}\ t^p\ ln\left|\frac{t}{a}\right| .$$

$$(viii)\quad f_0(t) = \left(\frac{e_{10}}{l-n}\right) t^l + \left(\frac{e_{11}}{m-n}\right) t^m + \left\{\frac{e_{12}}{p-n}\ ln\left|\frac{t}{a}\right| - \frac{e_{12}}{(p-n)^2}\right\}\ t^p + \left\{\frac{\bar{x}}{a^n} - \frac{e_{10} a^{l-n}}{l-n} - \frac{e_{11} a^{m-n}}{m-n} + \frac{x_0}{a^n(m-l)(m-n)^2}\right\} t^n ,$$

where

$$e_{10} = \frac{x_4}{a^l} - \frac{x}{a^l(m-l)} + \frac{x_0}{a^l(m-l)^2},\quad e_{11} = \frac{x}{a^m(m-l)} - \frac{x_0}{a^m(m-l)^2}\ and\quad e_{12} = \frac{x_0}{a^p(p-l)}.$$

$$(ix) f_0(t) = \left(\frac{e_7}{l-n} - \frac{e_9}{p-n}\right) t^l + \left\{\frac{\bar{x}}{a^n} - \frac{e_7\ a^{l-n}}{l-n} + \frac{e_9\ a^{p-n}}{(p-n)^2} + e_8 ln\left|\frac{t}{a}\right|\right\} t^n + \frac{x_0}{a^p(p-m)(p-n)}\ t^p\ ln\left|\frac{t}{a}\right| .$$

$$(x) f_0(t) = \left\{\frac{e_4}{l-n} - \frac{e_5}{(l-n)^2} + \frac{e_5}{l-n} ln\left|\frac{t}{a}\right|\right\} t^l + \left\{\frac{\bar{x}}{a^n} - \frac{e_4\ a^{l-n}}{l-n} + \frac{e_5\ a^{l-n}}{(l-n)^2}\right\} t^n + \frac{x_0}{a^p(p-l)^2}\ t^p\ ln\left|\frac{t}{a}\right| .$$

$$(xi) f_0(t) = \left\{\frac{e_{11}}{m-n} - \frac{e_{12}}{(m-n)^2}\right\} t^m + \left\{\frac{\bar{x}}{a^n} - \frac{e_{11}\ a^{m-n}}{m-n} + \frac{e_{12}\ a^{m-n}}{(m-n)^2} + e_{10}\ ln\left|\frac{t}{a}\right|\right\} t^n + \frac{x_0}{a^p(p-l)(p-n)}\ t^p\ ln\left|\frac{t}{a}\right| .$$

$$(xii) f_0(t) = \left\{\frac{x_5}{a^l(l-n)} - \frac{x}{a^l(l-n)^2} + \frac{x}{a^l(l-n)} ln\left|\frac{t}{a}\right|\right\} t^l + \left\{\frac{\bar{x}}{a^n} - \frac{x_5}{a^n(l-n)} + \frac{x}{a^n(l-n)^2} - \frac{x_0}{a^n(l-n)^3}\right\} t^n + \left\{\frac{x_0}{a^p(p-n)^3} - \frac{x_0}{a^p(p-n)^2} ln\left|\frac{t}{a}\right| + \frac{x_0}{2a^p(p-n)}\left(ln\left|\frac{t}{a}\right|\right)^2\right\}\ t^p .$$

$$(xiii) f_0(t) = \left(\frac{e_{10}}{l-n}\right) t^l + \left\{\frac{\bar{x}}{a^n} - \frac{e_{10}\ a^{l-n}}{l-n} + e_{11}\ ln\left|\frac{t}{a}\right|\right\} t^n + \frac{x_0}{2a^p(p-l)} t^p \left(ln\left|\frac{t}{a}\right|\right)^2 .$$

$$(xiv) f_0(t) = \left(\frac{e_8}{m-n}\right) t^m + + \left\{ \frac{\bar{x}}{a^n} - \frac{e_8\ a^{m-n}}{m-n} + e_7\ ln \left|\frac{t}{a}\right| \right\} t^n + + \frac{x_0}{2a^p(p-m)} t^p \left(ln \left|\frac{t}{a}\right| \right)^2 ,$$

and

$$(xv) f_0(t) = \left\{ \frac{\bar{x}}{a^l} + \frac{x}{a^l}\ ln \left|\frac{t}{a}\right| + + \frac{x}{2a^l} \left(ln \left|\frac{t}{a}\right| \right)^2 + \frac{x_0}{6a^l} \left(ln \left|\frac{t}{a}\right| \right)^3 \right\} t^l ,$$

where $x, \bar{x}, x_0, x_1, x_2, x_3, x_4$ and x_5 are limit points in X. These are the unique solutions of all possible cases. In fact, all these are the possible solutions of (1.4). Hence the theorem is proved. □

Example 3.1. *Let $X = \mathbb{R}$ be a Banach space and $I = (a, \infty)$, $a > 0$. Consider the Euler's equation*

(3.5)

$$t^4 y^{(iv)}(t) + 2t^2 y^{''}(t) - 4ty^{'}(t) + 4y(t) = 0$$

If we compare (3.5) with (1.4), then $\alpha = 0, \beta = 2, \gamma = -4$ and $\delta = 4$, and $l = 1, m = 1, n = 2, p = 2$ are the characteristic roots of (3.5). Let $f : I \to X$ satisfy the differential inequality

$$\left\| t^4 f^{(iv)}(t) + 2f^2 y^{''}(t) - 4tf^{'}(t) + 4f(t) \right\| \leq \epsilon$$

for any $\epsilon > 0$ and for any $t \in I$. Then, by Theorem 3.1, *there exists a unique solution $f_0 : I \to X$ such that*

$$\|f(t) - f_0(t)\| \leq |t^2| \left| \int_t^b u^{-3}\ \theta(u) du \right| ,$$

where $I = (a, b)$ and

$$\phi(t) = |t^2| \left| \int_t^b u^{-3}\ \epsilon du \right| = \frac{\epsilon}{2} \left| \left(\frac{t}{b}\right)^2 - 1 \right| .$$

When $b \to \infty, \phi(t) = \frac{\epsilon}{2}$. Also

$$\psi(t) = |t| \left| \int_t^b u^{-2}\ \frac{\epsilon}{2} du \right| = \frac{\epsilon}{2} \left| \left(\frac{t}{b}\right) - 1 \right| = \frac{\epsilon}{2}$$

and

$$\theta(t) = |t| \left| \int_t^b u^{-2}\ \frac{\epsilon}{2} du \right| = \frac{\epsilon}{2}$$

as $b \to \infty$. Hence,

$$\|f(t) - f_0(t)\| \leq |t^2| \left| \int_t^b u^{-3}\ \frac{\epsilon}{2} du \right| = \frac{\epsilon}{4} \left| \left(\frac{t}{b}\right)^2 - 1 \right| .$$

When $b \to \infty$,

$$\|f(t) - f_0(t)\| \leq \frac{\epsilon}{4},$$

where

$$f_0(t) = \left[\left(\frac{2x_0 - x - x_2}{a} \right) + \left(\frac{x_0 - x_2}{a} \right) ln \left|\frac{t}{a}\right| \right] t + \left[\left(\frac{\bar{x} + x + x_2 - 2x_0}{a^2} \right) + \left(\frac{x_0}{a^2} \right) ln \left|\frac{t}{a}\right| \right] t^2 ,$$

and $x, \bar{x}, x_0, x_2$ are the unique elements of X.

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