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STABILITY FOR CERTAIN CLASS OF MULTIVALENT FUNCTIONS

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Abstract. In this paper we investigate the problem of stability for a certain class of p-valent functions in T_δ-neighborhoods and we find the lower and upper bounds of radius of stability.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \ (p \in \mathbb{N} = \{1, 2, 3, ...\}),$$

which are analytic and p-valent in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $\mathcal{S}^*$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}(1) = \mathcal{A}$ consisting of starlike and convex functions, respectively.
A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{M}_p(\alpha, \beta)$ if it satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha\left|\frac{zf'(z)}{f(z)} - p\right| + \beta,$$

where $\alpha \leq 0$ and $\beta > p$.

We note if $f(z) \in \mathcal{M}_p(\alpha, \beta)$, then, for $\alpha < -1$, $\frac{zf'(z)}{f(z)}$ lies in the region $G_p \equiv G_p(\alpha, \beta) = \{w = u + iv : Rew < \alpha|w - p| + \beta\}$, that is, part of the complex plane which contains $w = p$ and is bounded by the ellipse

$$\left(u - \frac{p\alpha^2 - \beta}{\alpha^2 - 1}\right)^2 + \frac{\alpha^2}{\alpha^2 - 1}v^2 = \frac{\alpha^2(\beta - p)^2}{(\alpha^2 - 1)^2},$$

with vertices at the points

$$\left(\frac{p\alpha^2 - \beta}{\alpha^2 - 1}, \frac{\beta - p}{\sqrt{\alpha^2 - 1}}\right), \left(\frac{p\alpha^2 - \beta}{\alpha^2 - 1}, \frac{p - \beta}{\sqrt{\alpha^2 - 1}}\right), \left(\frac{p\alpha + \beta}{\alpha + 1}, 0\right), \left(\frac{p\alpha - \beta}{\alpha - 1}, 0\right).$$

For $p = 1$, the class $\mathcal{M}_1(\alpha, \beta) = \mathcal{MD}(\alpha, \beta)$ was studied earlier by J. Nishiwaki and S. Owa [10]. Many subclasses of the class $\mathcal{M}_p(\alpha, \beta)$ were studied in earlier works [2, 11, 9, 12, 18].

Let the Hadamard product (or convolution) of two functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \ , \ g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

be given by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

And the integral convolution be given by

$$(f \otimes g)(z) = z^p + \sum_{n=p+1}^{\infty} \frac{a_n b_n}{n} z^n.$$

Also, note that if I_p denotes $I_p(z) = z^p$ then

$$f * I_p = I_p \text{ and } f \otimes I_p = I_p \ .$$

The convolution has the algebraic properties of ordinary multiplication. In convolution theory, the concept of duality is important. Many authors have used the powerful method of duality for study properties of analytic functions(for example, see [7, 8, 17]). The concept of duality in geometric function theory was stated by Ruscheweyh in the book [15]. Let $\mathcal{V}^*$ denote the dual set of $\mathcal{V} \subset \mathcal{A}(p)$. Then

$$\mathcal{V}^* = \left\{g \in \mathcal{A}(p) : \frac{(f * g)(z)}{z^p} \neq 0, \forall f \in \mathcal{V}, \forall z \in \mathbb{U}\right\}.$$

For $p = 1$, we obtain the definition of dual set defined by Ruscheweyh [14]. Let $D \subset \mathcal{A}(p)$ be given such that $D^* = \mathcal{M}_p(\alpha, \beta)$. Then it is easy to see that

$$f \in \mathcal{M}_p(\alpha, \beta) \Longleftrightarrow \frac{(f * g)(z)}{z^p} \neq 0, \ (g \in D, z \in \mathbb{U}).$$

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If $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$, then T_δ-neighborhood of the function f is defined as

$$TN_\delta(f)=\left\{g(z)=z^p+\sum_{n=p+1}^{\infty}b_nz^n\in\mathcal{A}(p):\right.$$
$$\left.\sum_{n=p+1}^{\infty}T_n|a_n-b_n|\le\delta\right\},$$

where $\delta>0$ and $T=\{T_n\}_{n=p+1}^{\infty}$ is a sequence of positive numbers.

In [16, 6] authors investigated T_δ-neighborhood for various subclasses of analytic functions.
We also define $TN_\delta(A)=\bigcup_{f\in A}TN_\delta(f)$, $(A\subset\mathcal{A})$. For $p=1$, St. Ruscheweyh in [13] considered $T=\{n\}_{n=2}^{\infty}$ and showed that if $f\in\mathcal{K}$, then $TN_{1/4}(f)\subset\mathcal{S}^*$.

Assume that A,B are subclasses of the class $\mathcal{A}$. Then the set of all functions $f*g$ and $f\otimes g$, where $f\in A$ and $g\in B$, will be denoted by $A*B$ and $A\otimes B$, respectively. Let $A*B\subset C$, the Hadamard product is called T-C-stabel on the pair of classes (A,B) if there exists $\delta>0$ such that $TN_\delta(A)*TN_\delta(B)\subset C$. Stability of the integral convolution is defined in a similar way. The constant δ_T which characterizes the stability of Hadamard or integral convolution is called the radius of stability and it is defined as follows.

Definition 1.1. *Let A, B, C be the subclasses of the class $\mathcal{A}$ and $A*B\subset C$. Then a constant $\delta_T(A*B,C)$, such that*

$$\delta_T(A*B,C)=\sup\{\delta:TN_\delta(A)*TN_\delta(B)\subset C\},$$

is called the radius of stability of the convolution on the pair $(A,\ B)$. The constant $\delta_T(A\otimes B,C)$, such that

$$\delta_T(A\otimes B,C)=\sup\{\delta:TN_\delta(A)\otimes TN_\delta(B)\subset C\},$$

is called the radius of stability of the integral convolution on the pair $(A,\ B)$.

Bebnarz in [3] studied T-C-stability for certain classes of analytic functions. Also, Bednarz, Kanas, Sokół and Aghalary et al.[4, 5, 1] recently investigated the problem of stability for various subclasses of analytic functions. In this paper we investigate the problem of stability for the class $\mathcal{M}_p(\alpha,\beta)$ in T_δ-neighborhoods and we find the lower and upper bounds of radius of stability.

2. Preliminaries

We shall require the following definitions and lemmas to prove our main results.

Lemma 2.1. *Let*

$$H_t(z)=\frac{z^p}{p(1-z)^2}\left[p+(1-p)z-\frac{Bz}{B-p}\right],$$

where

$$B=t\alpha+\beta\pm i\sqrt{t^2-(t\alpha+\beta-p)^2},$$
$$(t^2-(t\alpha+\beta-p)^2\ge0,\ \frac{\beta-p}{1-\alpha}\le t\le\frac{p-\beta}{1+\alpha},$$
$$\alpha<-1,\beta>p).$$

Then a function $f\in\mathcal{A}(p)$ is in $\mathcal{M}_p(\alpha,\beta)$ if and only if for all z in $\mathbb{U}$,

$$\frac{(f*H_t)(z)}{z^p}\neq0\ .$$

Proof. Let us assume that for $f\in\mathcal{A}(p)$, $\dfrac{(f*H_t)(z)}{z}\neq0$, $(z\in\mathbb{U})$. Then we have

$$\begin{aligned}\frac{(f*H_t)(z)}{z^p}&=\\&=\left\{\left(f(z)*\frac{pz^p+(1-p)z^{p+1}}{p(1-z)^2}\right)-\right.\\&\left.-\frac{B}{B-p}\left(f(z)*\frac{z^{p+1}}{p(1-z)^2}\right)\right\}/z^p\\&=\frac{\frac{zf'(z)}{p}-\frac{B}{B-p}\left(\frac{zf'(z)}{p}-f(z)\right)}{z^p}\\&=\frac{Bpf(z)-pzf'(z)}{p(B-p)z^p}\neq0\ ,\end{aligned}$$

or

$$\frac{zf'(z)}{f(z)}\neq B$$

Since boundary of region $G_p\equiv G_p(\alpha,\beta)$ can be taken as $B=t\alpha+\beta\pm i\sqrt{t^2-(t\alpha+\beta-p)^2}$ for any $\dfrac{\beta-p}{1-\alpha}\le t\le\dfrac{p-\beta}{1+\alpha}$, this means that $\dfrac{zf'(z)}{f(z)}$ lies completely either inside G_p or complement of G_p for all $z\in\mathbb{U}$. At $z=0$, $\dfrac{zf'(z)}{f(z)}=p\in G_p$ so that $\dfrac{zf'(z)}{f(z)}\subset G_p$ for all $z\in\mathbb{U}$,which shows that $f\in\mathcal{M}_p(\alpha,\beta)$. The converse part follows easily since all the steps can be retraced back. This completes the proof of lemma 2.1. ☐

In view of the definition of dual set and Lemma 2.1 we can easily obtain the following result.

Corollary 2.1. *Let*

$$D=\{h\in\mathcal{A}:\ h(z)=\frac{z^p}{p(1-z)^2}\left[p+(1-p)z-\frac{Bz}{B-p}\right]\right\},$$

where

$$B = t\alpha + \beta \pm i\sqrt{t^2 - (t\alpha + \beta - p)^2},$$
$$(t^2 - (t\alpha + \beta - p)^2 \geq 0,$$
$$\frac{\beta - p}{1-\alpha} \leq t \leq \frac{p-\beta}{1+\alpha}, \alpha < -1, \beta > p).$$

Then $D^* = \mathcal{M}_p(\alpha, \beta)$.

Lemma 2.2. *Let* $\alpha < -1$ *and* $\beta > p$. *If* $h(z) = z^p + \sum_{n=p+1}^{\infty} c_n z^n \in D$. *Then*

$$|c_n| \leq \frac{\beta - p + (n-p)(1-\alpha)}{\beta - p}.$$

Proof. From the power series of the function $h(z) \in D$ in Corollary 2.1 we obtain

$$c_n = \frac{B-n}{B-p},$$

and therefore

$$|c_n|^2 = \frac{t^2 + (n-p)[n+p-2(t\alpha+\beta)]}{t^2}.$$

Since $\frac{\beta - p}{1-\alpha} \leq t \leq \frac{p-\beta}{1+\alpha}$, then $-2(t\alpha+\beta) \leq 2(t-p)$ and we get

$$\begin{aligned} |c_n|^2 &\leq \frac{t^2 + (n-p)[n+p+2(t-p)]}{t^2} \\ &= \frac{t^2 + (n-p)(n-p+2t)}{t^2} \\ &= \frac{(n-p+t)^2}{t^2}, \end{aligned}$$

and so

$$|c_n| \leq \frac{n-p+t}{t}.$$

Now, since $\frac{\beta - p}{1-\alpha} \leq t$, we obtain

$$\begin{aligned} |c_n| &\leq \frac{n-p+t}{t} \\ &= 1 + \frac{n-p}{t} \\ &\leq \frac{\beta - p + (n-p)(1-\alpha)}{\beta - p}. \end{aligned}$$

□

Corollary 2.2. *Let* $\alpha < -1$ *and* $\beta > p$. *The function* $g(z) = z^p + Az^n \in \mathcal{M}_p(\alpha, \beta)$ *if and only if*

$$|A| \leq \frac{\beta - p}{\beta - p + (n-p)(1-\alpha)}. \tag{2.1}$$

Proof. First we prove the sufficient condition. Since

$$\begin{aligned} \left|\frac{(g*h)(z)}{z^p}\right| &= \left|1 + c_n A z^{n-p}\right| \\ &\geq 1 - |c_n A z| \\ &\geq 1 - |z| > 0 \\ &(z \in \mathbb{U},\ h \in D) \end{aligned}$$

then by Corollary 2.1, $g \in D^* = \mathcal{M}_p(\alpha, \beta)$. Assume next, for neccessity, that $g \in \mathcal{M}_p(\alpha, \beta)$, and

$$h(z) = z^p + \sum_{n=p+1}^{\infty} \frac{\beta - p + (n-p)(1-\alpha)}{\beta - p} z^n \in D.$$

Then

$$\frac{(g*h)(z)}{z^p} = 1 + A\frac{\beta - p + (n-p)(1-\alpha)}{\beta - p} z^{n-p}.$$

Then, for $|A| > \frac{\beta - p}{\beta - p + (n-p)(1-\alpha)}$ there exist a point $\zeta \in \mathbb{U}$ such that $\frac{(g*h)(\zeta)}{\zeta^p} = 0$, so that the inequality (2.1) must hold. □

Corollary 2.3. *Let* $\alpha < -1, \beta > p$ *and* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \mathcal{A}(p)$. *If*

$$\sum_{n=p+1}^{\infty} \frac{\beta - p + (n-p)(1-\alpha)}{\beta - p} |a_n| \leq 1,$$

then $f \in \mathcal{M}_p(\alpha, \beta)$.

Proof. Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \mathcal{A}(p)$ and $h(z) = z^p + \sum_{n=p+1}^{\infty} c_n z^n \in D$. Since for all $n \geq 2$,

$$|c_n| \leq \frac{\beta - p + (n-p)(1-\alpha)}{\beta - p}.$$

Then we have

$$\begin{aligned} \left|\frac{(f*h)(z)}{z^p}\right| &= \left|1 + \sum_{n=p+1}^{\infty} a_n c_n z^{n-p}\right| \\ &\geq 1 - \sum_{n=p+1}^{\infty} |a_n||c_n||z| \\ &> 1 - \sum_{n=p+1}^{\infty} |a_n||c_n| \geq 0. \end{aligned}$$

Thus $\frac{(f*h)(z)}{z^p} \neq 0$ and from Corollary 2.1 we have $f \in D^* = \mathcal{M}_p(\alpha, \beta)$. □

Lemma 2.3. *Let* $\alpha < -1$ *and* $\beta > p$. *For* $f \in \mathcal{A}$ *and for every* $\epsilon \in \mathbb{C}$ *such that* $|\epsilon| < \delta$, *if* $F_\epsilon(z) = \frac{f(z) + \epsilon z^p}{1+\epsilon} \in \mathcal{M}_p(\alpha, \beta)$, *then for every* $h \in D, \left|\frac{(f*h)(z)}{z^p}\right| \geq \delta, (z \in \mathbb{U})$.

Proof. Let $F_\epsilon \in \mathcal{M}_p(\alpha, \beta)$. Then by Corollary 2.1, $\forall h \in D$, $\forall z \in \mathbb{U}$ $\frac{(F_\epsilon * h)(z)}{z^p} \neq 0$. Equivalently, $\frac{(f*h)(z) + \epsilon z^p}{(1+\epsilon)z^p} \neq 0$ in $\mathbb{U}$ or $\frac{(f*h)(z)}{z^p} \neq -\epsilon$ which shows that $\left|\frac{(f*h)(z)}{z^p}\right| \geq \delta$. □

Lemma 2.4. *([15]) Let* $f(z)$ *and* $g(z)$ *be in the class* $\mathcal{K}$ *and* $\mathcal{S}^*$ *respectively. Then, for every function* $F(z)$ *analytic in* $\mathbb{U}$, *we have*

$$\frac{f(z) * F(z)g(z)}{f(z) * g(z)} \in \overline{Co}(F(\mathbb{U})),\ z \in \mathbb{U},$$

where $\overline{Co}$ denotes the closed convex hull.

Definition 2.1.

$$\mathcal{K}^{(p)} = \left\{ f \in \mathcal{A}(p) : f(z) = z^{p-1}\Phi(z),\ \Phi \in \mathcal{K} \right\}.$$

Lemma 2.5. *Let $p < \beta \leq p-(1+\alpha)$ and $\alpha < -1$. If $f \in \mathcal{M}_p(\alpha,\beta)$ and $g \in \mathcal{K}^{(p)}$. Then $f * g \in \mathcal{M}_p(\alpha,\beta)$.*

Proof. Let $f \in \mathcal{M}(\alpha,\beta)$ and $g \in \mathcal{K}^{(p)}$. Then

$$\frac{zf'(z)}{f(z)} \in G_p\ ,\ g(z) = z^{p-1}\Phi(z),$$

where $\Phi(z) \in \mathcal{K}$. To prove the required result, it is sufficient to prove that

$$\frac{z(g*f)'(z)}{(g*f)(z)} \in G_p.$$

We have

$$\frac{z(z^{1-p}f(z))'}{z^{1-p}f(z)} = 1 - p + \frac{zf'(z)}{f(z)},$$

moreover, from propreties of region G_p we have

$$Re\frac{zf'(z)}{f(z)} > \frac{p\alpha+\beta}{\alpha+1}.$$

Then

$$Re\frac{z(z^{1-p}f(z))'}{z^{1-p}f(z)} > \frac{\beta - p + \alpha + 1}{\alpha+1} \geq 0.$$

Thus, $z^{1-p}f(z) \in \mathcal{S}^*$. Now we have

$$\begin{aligned}\frac{z(g*f)'(z)}{(g*f)(z)} &= \frac{g(z) * zf'(z)}{g(z) * f(z)} \\ &= \frac{z^{p-1}\Phi(z) * zf'(z)}{z^{p-1}\Phi(z) * f(z)} \\ &= \frac{\Phi(z) * z^{2-p}f'(z)}{\Phi(z) * z^{1-p}f(z)} \\ &= \frac{\Phi(z) * \dfrac{zf'(z)}{f(z)} z^{1-p}f(z)}{\Phi(z) * z^{1-p}f(z)}.\end{aligned}$$

Then by using Lemma 2.4, we have

$$\begin{aligned}\frac{z(g*f)'(z)}{(g*f)(z)} &= \frac{\Phi(z) * \dfrac{zf'(z)}{f(z)} z^{1-p}f(z)}{\Phi(z) * z^{1-p}f(z)} \\ &\in \overline{Co}(F(\mathbb{U})) \subset G_p,\end{aligned}$$

where $F(z) = \dfrac{zf'(z)}{f(z)}$ and G_p is a convex region. Then $f * g \in \mathcal{M}_p(\alpha,\beta)$. □

Definition 2.2. *A function $f \in \mathcal{A}(p)$ is said to be in the class $\mathcal{N}_p(\alpha,\beta)$ if for all $z \in \mathbb{U}$,*

$$zf'(z) + (1-p)f(z) \in \mathcal{M}_p(\alpha,\beta).$$

Lemma 2.6. *Let $p < \beta \leq p - (1+\alpha)$ and $\alpha < -1$. If $f \in \mathcal{N}_p(\alpha,\beta)$, then for ϵ with $|\epsilon| < \dfrac{1}{4}$,*

$$F_\epsilon(z) = \frac{f(z) + \epsilon z^p}{1+\epsilon} \in \mathcal{M}_p(\alpha,\beta).$$

Proof. Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$. Then

$$\begin{aligned}F_\epsilon(z) &= \frac{f(z)+\epsilon z^p}{1+\epsilon} \\ &= \frac{z^p(1+\epsilon) + \sum_{n=p+1}^{\infty} a_n z^n}{1+\epsilon} \\ &= \frac{f(z) * [z^p(1+\epsilon) + \sum_{n=p+1}^{\infty} z^n]}{1+\epsilon} \\ &= f(z) * \frac{z^p - \dfrac{\epsilon}{1+\epsilon} z^{p+1}}{1-z} \\ &= f(z) * k(z),\end{aligned}$$

where

$$\begin{aligned}k(z) &= \frac{z^p - \dfrac{\epsilon}{1+\epsilon} z^{p+1}}{1-z} \\ &= z^{p-1}h(z),\end{aligned}$$

and $h(z)\dfrac{z - \dfrac{\epsilon}{1+\epsilon}z^2}{1-z}$. Now,

$$\begin{aligned}\frac{zh'(z)}{h(z)} &= \frac{z - \frac{2\epsilon}{1+\epsilon}z^2}{z - \frac{\epsilon}{1+\epsilon}z^2} + \frac{z}{1-z} \\ &= \frac{-\rho z}{1-\rho z} + \frac{1}{1-z},\ \text{where } \rho = \frac{\epsilon}{1+\epsilon}.\end{aligned}$$

Hence $|\rho| < \dfrac{|\epsilon|}{1-|\epsilon|} < \dfrac{1}{3}$ gives $|\epsilon| < \dfrac{1}{4}$. Thus

$$Re\left(\frac{zh'(z)}{h(z)}\right) \geq \frac{1 - 2|\rho||z| - |\rho||z|^2}{(1-|\rho||z|)(1+|z|)} > 0,$$

if $|\rho|(|z|^2 + 2|z|) - 1 < 0$. This inequality holds for all $|\rho| < \dfrac{1}{3}$ and $|z| < 1$, which is true for $|\epsilon| < \dfrac{1}{4}$. Therefore h is starlike in $\mathbb{U}$ and so

$$\int_0^z \frac{h(t)}{t}dt = z + \sum_{n=2}^{\infty} \frac{h_n z^n}{n} = h(z) * \log\left(\frac{1}{1-z}\right),$$

is convex for $|\epsilon| < \dfrac{1}{4}$.
Also we have

$$\begin{aligned}(f*k)(z) = (k*f)(z) &= z^{p-1}h(z) * f(z) \\ &= z^{p-1}\left[h(z) * z^{1-p}f(z)\right] \\ &= z^{p-1}\left[h(z) * F(z)\right] \\ &= z^{p-1}\left[h(z) * \left(zF'(z) * \log\left(\frac{1}{1-z}\right)\right)\right] \\ &= z^{p-1}\left[zF'(z) * \left(h(z) * \log\left(\frac{1}{1-z}\right)\right)\right] \\ &= z^{p-1}\left[zF'(z) * \Psi(z)\right] \\ &= z^{p-1}\left[z^{1-p}(zf'(z) + (1-p)f(z)) * \Psi(z)\right] \\ &= (zf'(z) + (1-p)f(z)) * z^{p-1}\Psi(z),\end{aligned}$$

where $F(z) = z^{1-p}f(z)$ and $\Psi(z) = h(z) * \log\left(\dfrac{1}{1-z}\right)$. Now from $f \in \mathcal{N}_p(\alpha,\beta)$, we have

$zf'(z) + (1-p)f(z) \in \mathcal{M}_p(\alpha, \beta)$ and since $\Psi(z) \in \mathcal{K}$ then by Lemma 2.5 we have

$$(zf'(z) + (1-p)f(z)) * z^{p-1}\Psi(z) \in \mathcal{M}_p(\alpha, \beta).$$

Thus $F_\epsilon(z) = (f * k)(z) \in \mathcal{M}_p(\alpha, \beta)$. □

Lemma 2.7. *Let $p < \beta \le p - (1+\alpha)$ and $\alpha < -1$. If $f \in \mathcal{N}_p(\alpha, \beta)$ and $h \in D$, then $\left|\frac{(f*h)(z)}{z^p}\right| \ge \frac{1}{4}$.*

Proof. Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \mathcal{N}_p(\alpha, \beta)$ and $h \in D$, then from Lemma 2.6 for $|\epsilon| < \frac{1}{4}$ we have $F_\epsilon(z) = \frac{f(z) + \epsilon z^p}{1+\epsilon} \in \mathcal{M}_p(\alpha, \beta)$. Thus

$$\frac{1}{z^p}[h(z) * F_\epsilon(z)] \neq 0, \ |\epsilon| < \frac{1}{4}.$$

Now from the properties of Hadamard product we obtain

$$\frac{1+\epsilon}{z^p}[h(z) * \frac{f(z)+\epsilon z^p}{1+\epsilon}] = \frac{1}{z^p}[h(z) * (f(z) + \epsilon z^p)] = \frac{1}{z^p}[h(z) * f(z)] + \epsilon \neq 0 \ .$$

Hence for $|\epsilon| < \frac{1}{4}$, $\frac{1}{z^p}[h(z) * f(z)] \neq -\epsilon$, and so $\left|\frac{(f*h)(z)}{z^p}\right| \ge \frac{1}{4}$. □

Lemma 2.8. *Let $\beta > p$ and $\alpha \le 0$. If $f \in \mathcal{M}_p(\alpha, \beta)$, then*

$$|a_{p+1}| \le \frac{2(\beta - p)}{1-\alpha},$$

and

$$|a_n| \le \frac{2(\beta-p)}{(n-p)(1-\alpha)} \prod_{j=1}^{n-p-1} \left(1 + \frac{2(\beta-p)}{j(1-\alpha)}\right), \quad (n \ge p+2).$$

Proof. Let $f \in \mathcal{M}_p(\alpha, \beta)$. Then

$$Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha \left|\frac{zf'(z)}{f(z)} - p\right| + \beta \le \alpha \left(\frac{zf'(z)}{f(z)} - p\right) + \beta,$$

implies that

$$\beta - p\alpha + (\alpha - 1)Re\left(\frac{zf'(z)}{f(z)}\right) > 0.$$

And let us define the function $p(z)$ by

$$p(z) = \frac{\beta - p\alpha + (\alpha-1)\frac{zf'(z)}{f(z)}}{\beta - p}. \tag{2.2}$$

Then $p(z)$ is analytic in $\mathbb{U}$, $p(0) = 1$ and $Rep(z) > 0$ $(z \in \mathbb{U})$. Therefore, if we write

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{2.3}$$

then $|p_n| \le 2$ $(n \ge 1)$. From (2.2) and (2.3), we obtain that

$$(\alpha - 1) \sum_{n=p+1}^{\infty} (n-p)a_n z^n = (\beta - p)\sum_{n=1}^{\infty} p_n z^n (z^p + \sum_{n=p+1}^{\infty} a_n z^n).$$

Therefore we have

$$a_n = \frac{\beta - p}{(n-p)(\alpha-1)} \times (p_{n-p} + p_{n-p-1}a_{p+1} + ... + p_2 a_{n-2} + p_1 a_{n-1}),$$

for all $n \ge p+1$. When $n = p+1$,

$$|a_{p+1}| \le \frac{\beta-p}{1-\alpha}|p_1| \le \frac{2(\beta-p)}{1-\alpha}.$$

And when $n = p+2$,

$$|a_{p+2}| \le \frac{\beta-p}{2(1-\alpha)}(|p_2| + |p_1||a_{p+1}|) \le \frac{2(\beta-p)}{2(1-\alpha)}\left(1 + \frac{2(\beta-p)}{1-\alpha}\right).$$

Let us suppose that

$$|a_k| \le \frac{2(\beta-p)}{(k-p)(1-\alpha)} (1 + |a_{p+1}| + ... + |a_{k-2}| + |a_{k-1}|) \le \frac{2(\beta-p)}{(k-p)(1-\alpha)} \prod_{j=1}^{k-1-p}\left(1 + \frac{2(\beta-p)}{j(1-\alpha)}\right) \ (k \ge p+2). \tag{2.4}$$

Then we see

$$1 + |a_{p+1}| + ... + |a_{k-2}| + |a_{k-1}| \le \prod_{j=1}^{k-1-p}\left(1 + \frac{2(\beta-p)}{j(1-\alpha)}\right). \tag{2.5}$$

By using (2.4) and (2.5), we obtain that

$$|a_{k+1}| \le \frac{2(\beta-p)}{(k+1-p)(1-\alpha)} (1 + |a_{p+1}| + ... + |a_{k-2}| + |a_{k-1}| + |a_k|) \le \left(1 + \frac{2(\beta-p)}{(k-p)(1-\alpha)}\right) \frac{2(\beta-p)}{(k+1-p)(1-\alpha)} \prod_{j=1}^{k-1-p}\left(1 + \frac{2(\beta-p)}{j(1-\alpha)}\right) \le \frac{2(\beta-p)}{(k+1-p)(1-\alpha)} \prod_{j=1}^{k-p}\left(1 + \frac{2(\beta-p)}{j(1-\alpha)}\right).$$

This completes the proof of the lemma 2.8. □

Corollary 2.4. *Let $\alpha \leq -1$ and $p < \beta \leq p+2$. If $f \in \mathcal{M}_p(\alpha, \beta)$, then*

$$|a_n| \leq \frac{n(\beta-p)}{2} \ (n \geq p+1).$$

Proof. Since $\alpha \leq -1$ and $p < \beta \leq p+2$, then $\frac{1}{1-\alpha} \leq \frac{1}{2}$, $0 < \beta - p \leq 2$ and from Lemma 2.8 for $n \geq p+2$ we obtain that

$$\begin{aligned} |a_n| &\leq \\ &\frac{2(\beta-p)}{(n-p)(1-\alpha)} \prod_{j=1}^{n-p-1} \left(1 + \frac{2(\beta-p)}{j(1-\alpha)}\right) \\ &\leq \frac{\beta-p}{n-p} \prod_{j=1}^{n-p-1} \left(1 + \frac{\beta-p}{j}\right) \\ &= \frac{\beta-p}{n-p} \times \\ &(\frac{(1+\beta-p)(2+\beta-p)...(n-p-1+\beta-p)}{1 \times 2 \times ... \times (n-p-1)} \\ &= \frac{(\beta-p)(\beta-p+1)\cdots(\beta-p+n-p-1)}{(n-p)!} \\ &\leq (\beta-p)\frac{2 \times 3 \times 4 \times ... \times (n-p) \times (n-p+1)}{2(n-p)!} \\ &= \frac{(\beta-p)(n-p+1)}{2} \\ &\leq \frac{n(\beta-p)}{2}. \end{aligned}$$

Also, for $n = p+1$ again from Lemma 2.8 we have

$$\begin{aligned} |a_{p+1}| &\leq \frac{2(\beta-p)}{1-\alpha} \\ &\leq \beta - p \\ &\leq \frac{(p+1)(\beta-p)}{2}. \end{aligned}$$

□

Lemma 2.9. *Let $L_\gamma : \mathcal{A} \to \mathcal{A}$ is the integral operator defined by*

$$L_\gamma[f](z) = \frac{\gamma+1}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt,$$

and $Re\gamma \geq 0$, then $L_\gamma[\mathcal{K}] \subset \mathcal{K}$.

Corollary 2.5. *Let $1 < p \leq 3$, then*

$$g(z) = z + \sum_{n=2}^{\infty} \frac{n}{p+n-1} z^n \in \mathcal{K}.$$

Proof. Let $1 < p \leq 3$ and $\gamma = \frac{n^2-n+1-p}{p-1}$, $n \geq 2$. Since $f(z) = z + \sum_{n=2}^\infty z^n \in \mathcal{K}$ and $\gamma \geq 0$, then by Lemma 2.9 we have

$$L_\gamma[f](z) = g(z) = z + \sum_{n=2}^{\infty} \frac{n}{p+n-1} z^n \in \mathcal{K}.$$

□

3. Main results

Throughout this section $T = \{T_n\}_{n=p+1}^\infty$ will always be the sequence given by

$$T_n = \frac{\beta - p + (n-p)(1-\alpha)}{\beta - p},$$

unless otherwise mentioned.

Theorem 3.1. *Let $1 \leq p \leq 3$ and*

$$\begin{cases} p < \beta \leq p - (\alpha+1) & -3 \leq \alpha < -1, \\ p < \beta \leq p+2 & \alpha < -3. \end{cases}$$

Then for

$$0 \leq \delta < \sqrt{\left[\frac{(\beta-p+1-\alpha)(p+3)}{4(\beta-p)}\right]^2 + \frac{(p+1)(\beta-p+1-\alpha)}{4(\beta-p)}} - \left[\frac{(\beta-p+1-\alpha)(p+3)}{4(\beta-p)}\right],$$

we have

$$TN_\delta(\mathcal{M}_p(\alpha,\beta)) \otimes TN_\delta(\mathcal{K}^{(p)}) \subset \mathcal{M}_p(\alpha, \beta).$$

Proof. Let $f_0(z) = z^p + \sum_{n=p+1}^\infty a_{0n} z^n \in \mathcal{M}_p(\alpha,\beta)$ and $g_0(z) = z^p + \sum_{n=p+1}^\infty b_{0n} z^n \in \mathcal{K}^{(p)}$. Then $g_0(z) = z^{p-1}\Phi(z)$ where $\Phi(z) = z + \sum_{k=2}^\infty d_k z^k \in \mathcal{K}$ and so we have $g_0(z) = z^p + \sum_{n=p+1}^\infty d_{n-p+1} z^n$. Consequently, $b_{0n} = d_m$ where $m = n-p+1$ for all $n \geq p+1$. Also suppose that $f(z) = z^p + \sum_{n=p+1}^\infty a_n z^n \in TN_\delta(f_0)$ and $g(z) = z^p + \sum_{n=p+1}^\infty b_n z^n \in TN_\delta(g_0)$. We want to show that

$$\frac{(f \otimes g * h)(z)}{z^p} \neq 0, \ (h \in D).$$

By the identity

$$\begin{aligned} &f \otimes g * h = \\ &f_0 \otimes g_0 * h + f_0 \otimes (g - g_0) * h + \\ &(f - f_0) \otimes g_0 * h + (f-f_0) \otimes (g-g_0) * h, \end{aligned}$$

we obtain

$$\left|\frac{(f \otimes g * h)(z)}{z^p}\right| \geq \tag{3.1}$$

$$\geq \left|\frac{(f_0 \otimes g_0 * h)(z)}{z^p}\right| - \left|\frac{(f_0 \otimes (g-g_0) * h)(z)}{z^p}\right| - \left|\frac{((f-f_0) \otimes g_0 * h)(z)}{z^p}\right| - \left|\frac{((f-f_0)\otimes(g-g_0)*h)(z)}{z^p}\right|.$$

From Lemma 2.5 we have $(f_0 * g_0)(z) \in \mathcal{M}_p(\alpha,\beta)$. We have

$$z(f_0 \otimes g_0)'(z) + (1-p)(f_0 \otimes g_0)(z) = (f_0 * g_0)(z) * (z^p + \sum_{n=p+1}^\infty \frac{n+1-p}{n} z^n),$$

and

$$z^p + \sum_{n=p+1}^\infty \frac{n+1-p}{n} z^n = z^{p-1}(z + \sum_{l=2}^\infty \frac{l}{p+l-1} z^l).$$

Then clearly, for $p = 1$ and by using Lemma 2.5 and Corollary 2.5 for $1 < p \le 3$ we conclude that $z(f_0 \otimes g_0)'(z) + (1-p)(f_0 \otimes g_0)(z) \in \mathcal{M}_p(\alpha, \beta)$ and thus $f_0 \otimes g_0 \in \mathcal{N}_p(\alpha, \beta)$. Now from Lemma 2.7 we obtain

$$\left|\frac{(f_0 \otimes g_0 * h)(z)}{z^p}\right| \ge \frac{1}{4}. \tag{3.2}$$

Moreover, since $\Phi(z) = z + \sum_{k=2}^{\infty} d_k z^k \in \mathcal{K}$ and $b_{0n} = d_m$ where $m = n-p+1$ for all $n \ge p+1$ and also for all $n \ge p+1$, $m \ge 2$ therefore $|b_{0n}| = |d_m| \le 1$. Also by making use of Corollary 2.4 and Lemma 2.2 for $h(z) = z^p + \sum_{n=p+1}^{\infty} c_n z^n$ we obtain $|a_{0n}| \le \dfrac{n(\beta-p)}{2}$ and $|c_n| \le \dfrac{\beta - p + (n-p)(1-\alpha)}{\beta - p}$, respectively. Now, from the definitions of $TN_\delta(f_0)$ and $TN_\delta(g_0)$ we have

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} \frac{|a_{0n}||b_n - b_{0n}||c_n|}{n} \\
&\le \frac{\beta-p}{2} \sum_{n=p+1}^{\infty} \frac{\beta - p + (n-p)(1-\alpha)}{\beta-p} |b_n - b_{0n}| \\
&= \frac{\beta-p}{2} \sum_{n=p+1}^{\infty} T_n |b_n - b_{0n}| \\
&\le \frac{\delta(\beta-p)}{2}.
\end{aligned}$$

So

$$\sum_{n=2}^{\infty} \frac{|a_{0n}||b_n - b_{0n}||c_n|}{n} \le \frac{\delta(\beta-p)}{2}. \tag{3.3}$$

Similarly, we get

$$\begin{aligned}
\sum_{n=p+1}^{\infty} \frac{|b_{0n}||a_n - a_{0n}||c_n|}{n} &\le \frac{1}{p+1} \sum_{n=p+1}^{\infty} T_n |a_n - a_{0n}| \\
&\le \frac{\delta}{p+1}. \qquad (3.4)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} \frac{|a_n - a_{0n}||b_n - b_{0n}||c_n|}{n} \\
&\le \frac{\delta(\beta-p)}{(p+1)(\beta-p+1-\alpha)} \sum_{n=p+1}^{\infty} T_n |b_n - b_{0n}| \\
&\le \frac{\delta^2(\beta-p)}{(p+1)(\beta-p+1-\alpha)}. \qquad (3.5)
\end{aligned}$$

By virtue of (3.2),(3.3),(3.4) and (3.5), inequality (3.1) gives

$$\left|\frac{(f \otimes g * h)(z)}{z^p}\right| \ge \frac{1}{4} - \frac{\delta(\beta-p)}{2} - \frac{\delta}{p+1} - \frac{\delta^2(\beta-p)}{(p+1)(\beta-p+1-\alpha)}. \tag{3.6}$$

The right side of (3.6) is positive whenever

$$0 \le \delta < \sqrt{\left[\frac{(\beta-p+1-\alpha)(p+3)}{4(\beta-p)}\right]^2 + \frac{(p+1)(\beta-p+1-\alpha)}{4(\beta-p)}} - \left[\frac{(\beta-p+1-\alpha)(p+3)}{4(\beta-p)}\right].$$

$\square$

Corollary 3.1. *Let* $1 \le p \le 3$ *and*

$$\begin{cases} p < \beta \le p - (\alpha+1) & -3 \le \alpha < -1, \\ p < \beta \le p+2 & \alpha < -3. \end{cases}$$

Then we have

$$\delta_T(\mathcal{M}_p(\alpha,\beta) \otimes \mathcal{K}^{(p)}, \mathcal{M}_p(\alpha,\beta)) \ge \sqrt{\left[\frac{(\beta-p+1-\alpha)(p+3)}{4(\beta-p)}\right]^2 + \frac{(p+1)(\beta-p+1-\alpha)}{4(\beta-p)}} - \left[\frac{(\beta-p+1-\alpha)(p+3)}{4(\beta-p)}\right].$$

Theorem 3.2. *Let* $p < \beta \le p+2$ *and* $\alpha < -1$. *For*

$$0 \le \delta < \sqrt{\frac{[(p+1)(\beta-p+1-\alpha)]^2}{16} + \frac{(p+1)(\beta-p+1-\alpha)}{\beta-p}} - \frac{(p+1)(\beta-p+1-\alpha)}{4},$$

we have

$$TN_\delta(\{I_p\}) \otimes TN_\delta(\mathcal{M}_p(\alpha,\beta)) \subset \mathcal{M}_p(\alpha,\beta).$$

Proof. Let $f_0(z) = I_p(z) = z^p$ and $g_0(z) = z^p + \sum_{n=p+1}^{\infty} b_{0n} z^n \in \mathcal{M}_p(\alpha,\beta)$. Also suppose that $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in N_\delta(f_0)$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in N_\delta(g_0)$, then we have $\sum_{n=p+1}^{\infty} T_n |a_n| \le \delta$ and $\sum_{n=p+1}^{\infty} T_n |b_n - b_{0n}| \le \delta$. We want to show that

$$\frac{((f \otimes g) * h)(z)}{z^p} \ne 0, \ (h \in D).$$

We have

$$\begin{aligned}
&f \otimes g * h = \\
&f_0 \otimes g_0 * h + f_0 \otimes (g - g_0) * h \\
&\quad + (f - f_0) \otimes g_0 * h + (f - f_0) \otimes (g - g_0) * h.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\left|\frac{(f \otimes g * h)(z)}{z^p}\right| \ge \\
&\ge \left|\frac{(f_0 \otimes g_0 * h)(z)}{z^p}\right| - \left|\frac{(f_0 \otimes (g-g_0) * h)(z)}{z^p}\right| \\
&- \left|\frac{((f-f_0) \otimes g_0 * h)(z)}{z^p}\right| - \left|\frac{((f-f_0) \otimes (g-g_0) * h)(z)}{z^p}\right|.
\end{aligned}$$

Observe that, $(f_0 \otimes g_0 * h)(z) = z^p$ and $(f_0 \otimes (g - g_0) * h)(z) = 0$. Moreover we have

$$\begin{aligned}
&\left|\frac{((f-f_0)\otimes g_0 * h)(z)}{z^p}\right| \\
&\leq \sum_{n=p+1}^{\infty} \frac{|a_n||b_{0n}||c_n|}{n} \\
&\leq \frac{\beta-p}{2}\sum_{n=p+1}^{\infty} \frac{\beta-p+(n-p)(1-\alpha)}{\beta-p}|a_n| \\
&= \frac{\beta-p}{2}\sum_{n=p+1}^{\infty} T_n|a_n| \\
&\leq \frac{(\beta-p)\delta}{2},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{((f-f_0)\otimes (g-g_0) * h)(z)}{z^p}\right| \\
&\leq \sum_{n=p+1}^{\infty} \frac{|a_n||b_n-b_{0n}||c_n|}{n} \\
&\leq \frac{1}{p+1}\sum_{n=p+1}^{\infty} T_n|a_n||b_n-b_{0n}| \\
&\leq \frac{\delta(\beta-p)}{(p+1)(\beta-p+1-\alpha)}\sum_{n=p+1}^{\infty} T_n|b_n-b_{0n}| \\
&\leq \frac{\delta^2(\beta-p)}{(p+1)(\beta-p+1-\alpha)}.
\end{aligned}$$

Now, following the same techniques as in the proof of Theorem 3.1 we conclude the result and we omit details. □

Corollary 3.2. *Let $p < \beta \leq p+2$ and $\alpha < -1$. Then we have*

$$\delta_T(\{I_p\} \otimes \mathcal{M}_p(\alpha,\beta), \mathcal{M}_p(\alpha,\beta)) \geq \sqrt{\frac{[(p+1)(\beta-p+1-\alpha)]^2}{16} + \frac{(p+1)(\beta-p+1-\alpha)}{\beta-p}} - \frac{(p+1)(\beta-p+1-\alpha)}{4}.$$

Corollary 3.3. *Let $\beta > p$ and $\alpha < -1$. Then we have*

$$\delta_T(\mathcal{M}_p(\alpha,\beta) \otimes \mathcal{K}^{(p)}, \mathcal{M}_p(\alpha,\beta)) \leq$$

$$\mu_1 = \sqrt{\left[1+\frac{1-\alpha}{2(\beta-p)})\right]^2 + \frac{p(\beta-p+1-\alpha)}{\beta-p}} - \left[1+\frac{1-\alpha}{2(\beta-p)}\right], \tag{3.7}$$

and

$$\delta_T(\{I_p\} \otimes \mathcal{M}_p(\alpha,\beta), \mathcal{M}_p(\alpha,\beta)) \leq$$

$$\mu_2 = \sqrt{\frac{1}{4} + \frac{(p+1)(\beta-p+1-\alpha)}{\beta-p}} - \frac{1}{2}. \tag{3.8}$$

Proof. Let $g_0(z) = z^p + z^{p+1} + z^{p+2} + \dots$, $f_0(z) = z^p + \dfrac{\beta-p}{\beta-p+1-\alpha}z^{p+1}$, $h_0(z) = I_p(z) = z^p$. Since $g_0(z) = z^{p-1}(z + z^2 + z^3 + \dots = z^{p-1}(\dfrac{z}{1-z})$ and $\dfrac{z}{1-z} \in \mathcal{K}$ then $g_0 \in \mathcal{K}^{(p)}$. Also from Corollary 2.2, we have $f_0 \in \mathcal{M}_p(\alpha,\beta)$. Let

$$g(z) = z^p + \left(1+\frac{\delta(\beta-p)}{\beta-p+1-\alpha}\right)z^{p+1} + z^{p+2} + \dots \in TN_\delta(g_0) \subset TN_\delta(\mathcal{K}^{(p)}),$$

$$f(z) = z^p + \left(\frac{\beta-p}{\beta-p+1-\alpha}(1+\delta)\right)z^{p+1} \in TN_\delta(f_0) \subset TN_\delta(\mathcal{M}_p(\alpha,\beta)),$$

$$h(z) = z^p + \frac{\delta(\beta-p)}{\beta-p+1-\alpha}z^{p+1} \in TN_\delta(h_0) \subset TN_\delta(\{I_p\}).$$

To show (3.7) and (3.8) it is sufficient to prove that

$$f \otimes g \notin \mathcal{M}_p(\alpha,\beta) \text{ when } \delta > \mu_1,$$

and

$$h \otimes f \notin \mathcal{M}_p(\alpha,\beta) \text{ when } \delta > \mu_2.$$

We have

$$(f\otimes g)(z) = z^p + \frac{\beta-p}{(p+1)(\beta-p+1-\alpha)}(1+\delta)\left(1+\frac{\delta(\beta-p)}{\beta-p+1-\alpha}\right)z^{p+1}.$$

Let $\varphi(\delta) = \dfrac{\beta-p}{(p+1)(\beta-p+1-\alpha)}(1+\delta)\left(1+\frac{\delta(\beta-p)}{\beta-p+1-\alpha}\right)$. Then we have $\varphi(\mu_1) = \dfrac{\beta-p}{\beta-p+1-\alpha}$ and $\varphi(\delta) > \varphi(\mu_1)$ for $\delta > \mu_1$, therefore by Crollary 2.2 we have $(f\otimes g)(z) \notin \mathcal{M}_p(\alpha,\beta)$ when $\delta > \mu_1$.
Also we have

$$(h\otimes f)(z) = z^p + \frac{(\beta-p)^2}{(p+1)(\beta-p+1-\alpha)^2}\left(\delta+\delta^2\right)z^{p+1},$$

and

$$\frac{(\beta-p)^2}{(p+1)(\beta-p+1-\alpha)^2}\left(\mu_2+\mu_2^2\right) = \frac{\beta-p}{\beta-p+1-\alpha},$$

thus similarly $h\otimes f \notin \mathcal{M}_p(\alpha,\beta)$ when $\delta > \mu_2$. □

Theorem 3.3. *Let $\beta > p$ and $\alpha < -1$. Then we have*
(i) for

$$\delta_1 = \sqrt{\frac{\beta-p+1-\alpha}{\beta-p}},$$

$TN_{\delta_1}(\{I_p\}) * TN_{\delta_1}(\{I_p\}) \subset \mathcal{M}_p(\alpha,\beta)$,
(ii) for

$$\delta_2 = \sqrt{\frac{(p+1)(\beta-p+1-\alpha)}{\beta-p}},$$

$TN_{\delta_2}(\{I_p\}) \otimes TN_{\delta_2}(\{I_p\}) \subset \mathcal{M}_p(\alpha,\beta)$.
The result is the best possible in each case.

Proof. (i) Let

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in TN_{\delta_1}(\{I_p\})$$

and

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in TN_{\delta_1}(\{I_p\}).$$

By making use of definition $TN_{\delta_1}(\{I\})$ we have

$$\sum_{n=p+1}^{\infty} T_n|a_n| \le \delta_1, \tag{3.9}$$

and

$$\sum_{n=p+1}^{\infty} T_n|b_n| \le \delta_1. \tag{3.10}$$

Since $\beta > p$ and $\alpha < -1$, $T_n = \dfrac{\beta-p+(n-p)(1-\alpha)}{\beta-p}$ is an increasing function of n $(n \ge p+1)$ so that from (3.9) we get

$$\sum_{n=p+1}^{\infty} |a_n| \le \frac{(\beta-p)\delta_1}{\beta-p+1-\alpha},$$

which implies that

$$|a_n| \le \frac{(\beta-p)\delta_1}{\beta-p+1-\alpha} \ (n \ge p+1).$$

Using the above inequality and (3.10), it follows that

$$\sum_{n=p+1}^{\infty} \frac{\beta-p+(n-p)(1-\alpha)}{\beta-p}|a_n||b_n| \le \frac{(\beta-p)\delta_1^2}{\beta-p+1-\alpha} = 1,$$

which in view of Corollary 2.3, $(f*g)(z) \in \mathcal{M}_p(\alpha,\beta)$. The proof of (ii) is similar to part (i) and we omit the details.
To see that the containment relation in (i) is the best pssible, we consider the functions f and g defined in $\mathbb{U}$ by

$$f(z) = g(z) = z^p + \sqrt{\frac{\beta-p}{\beta-p+1-\alpha}}z^{p+1},$$

clearly, $f, g \in TN_{\delta_1}(I_p)$ and $(f*g) \in \mathcal{M}_p(\alpha,\beta)$. Also considering the functions f and g defined in $\mathbb{U}$ by

$$f(z) = g(z) = z^p + \sqrt{\frac{(p+1)(\beta-p)}{\beta-p+1-\alpha}}z^{p+1},$$

it is easily seen that the result in (ii) is the best possible. This evidently completes the proof. ☐

Corollary 3.4. *Let $\beta > p$ and $\alpha < -1$. Then we have*
(i)

$$\delta_T(\{I_P\} * \{I_p\}, \mathcal{M}_p(\alpha,\beta)) = \sqrt{\frac{\beta-p+1-\alpha}{\beta-p}},$$

(ii)

$$\delta_T(\{I_P\}\otimes\{I_p\}, \mathcal{M}_p(\alpha,\beta)) = \sqrt{\frac{(p+1)(\beta-p+1-\alpha)}{\beta-p}}.$$

Proof. (i) From Theorem 3.3 we have

$$\delta_T(\{I_P\} * \{I_p\}, \mathcal{M}_p(\alpha,\beta)) \ge \delta_1 = \sqrt{\frac{\beta-p+1-\alpha}{\beta-p}}. \tag{3.11}$$

Moreover, let

$$f(z) = g(z) = z^p + \frac{\delta(\beta-p)}{\beta-p+1-\alpha}z^{p+1} \in TN_\delta(\{I_p\}).$$

Then we have

$$(f*g)(z) = z^p + \left(\frac{\delta(\beta-p)}{\beta-p+1-\alpha}\right)^2 z^{p+1}.$$

Let $\varphi(\delta) = \left(\dfrac{\delta(\beta-p)}{\beta-p+1-\alpha}\right)^2$, then $\varphi(\delta) > \varphi(\delta_1) = \dfrac{\beta-p}{\beta-p+1-\alpha}$ for $\delta > \delta_1$. Therefore, by Corollary 2.2, $(f*g)(z) \notin \mathcal{M}_p(\alpha,\beta)$ when $\delta > \delta_1$. This means that

$$\delta_T(\{I_P\} * \{I_p\}, \mathcal{M}_p(\alpha,\beta)) \le \delta_1. \tag{3.12}$$

The relations (3.11) and (3.12) give the result. The proof of part (ii) is similar to part (i) and we omit the details. ☐

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