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SOME NEW RESULTS ASSOCIATED WITH CARATHÉODORY FUNCTIONS OF ORDER α

H. M. SRIVASTAVA AND SHIGEYOSHI OWA[1]

Abstract. Let $\mathcal{P}(\alpha)$ be the class of functions $p(z)$ which are Carathéodory functions of order α $(0 \leq \alpha < 1)$ in the open unit disk $\mathbf{U}$. Considering the extremal function $p(z)$ for the class $\mathcal{P}(\alpha)$, a new class $\mathcal{P}^*(\beta)$ $(\beta \in \mathbf{R})$ of functions $q(z)$ in $\mathbf{U}$ is defined. The object of the present paper is to develop several interesting coefficient inequalities for the functions $q(z)$ in the new class $\mathcal{P}^*(\beta)$ introduced here.

1. Introduction

Let $\mathcal{P}$ denote the class of functions $p(z)$ of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n ,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also, let $\mathcal{P}(\alpha)$ be the subclass of $\mathcal{P}$ consisting of functions $p(z)$ which satisfy the following inequality:

$$\Re\big(p(z)\big) > \alpha \qquad (z \in \mathbb{U})$$

for some real α $(0 \leq \alpha < 1)$. If $p(z) \in \mathcal{P}(0)$, then $p(z)$ is said to be a Carathéodory function in $\mathbb{U}$ (cf. [1] and [3]). Therefore, a function $p(z)$ in the class $\mathcal{P}(\alpha)$ is said to be a Carathéodory function of order α in $\mathbb{U}$ (see [4] and [9]). Furthermore, some interesting sufficient conditions for Carathéodory functions were recently investigated by Cho and Kim [2], Owa (see [5] and [6]), Shiraishi, Owa and Srivastava [7], and Sim, Kwon, Cho and Srivastava [8].

It is well known that a function $p(z)$ given by

$$p(z) = \frac{1 + (1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)\sum_{n=1}^{\infty} z^n \tag{1.1}$$

is the extremal function for the class $\mathcal{P}(\alpha)$. In view of this extremal function $p(z)$ given by (1.1), we consider a function $q(z)$ defined by

$$\begin{aligned} q(z) &= \frac{1 + (1-2\alpha)z}{1 - \sqrt[3]{z^2}} \\ &= 1 + \sum_{n=1}^{\infty} z^{\frac{2n}{3}} + (1-2\alpha)\sum_{n=1}^{\infty} z^{\frac{2n+1}{3}} \\ &= 1 + \left(1 + (1-2\alpha)z^{\frac{1}{3}}\right)\sum_{n=1}^{\infty} z^{\frac{2n}{3}} \quad (z \in \mathbb{U}), \end{aligned} \tag{1.2}$$

where we consider the *principal* value for $\sqrt[3]{z}$. For $q(z)$ given by (1.2), we see that $q(0) = 1$ and

$$\begin{aligned} \Re\left(q(z)\right) &= \Re\left(\frac{1 + (1-2\alpha)e^{i\theta}}{1 - e^{\frac{2}{3}i\theta}}\right) \\ &= \Re\left(\frac{e^{-\frac{1}{3}i\theta} + (1-2\alpha)e^{\frac{2}{3}i\theta}}{e^{-\frac{1}{3}i\theta} - e^{\frac{1}{3}i\theta}}\right) \\ &= \frac{1}{2} - (1-2\alpha)\cos\left(\frac{1}{3}\theta\right) \\ &> \begin{cases} \dfrac{4\alpha - 1}{2} & \left(\dfrac{1}{4} \leq \alpha < \dfrac{1}{2}\right) \\[2ex] \dfrac{3 - 4\alpha}{2} & \left(\dfrac{1}{2} \leq \alpha \leq \dfrac{3}{4}\right) \end{cases} \end{aligned}$$

for $z = e^{i\theta}$.

In view of the above considerations, we introduce a new idea for Carathéodory functions. Let $\mathcal{Q}$ be the

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[1] *corresponding author*

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class of functions $q(z)$ of the form:

$$q(z) = 1 + \sum_{n=1}^{\infty} \left(a_{\frac{2n}{3}} + a_{\frac{2n+1}{3}} z^{\frac{1}{3}} \right) z^{\frac{2n}{3}}, \tag{1.3}$$

which are analytic in $\mathbb{U}$. Here, as before, we consider the *principal* value for $z^{\frac{1}{3}}$. If $q(z) \in \mathcal{Q}$ satisfies the following inequality:

$$\Re\big(q(z)\big) > \beta \qquad (z \in \mathbb{U})$$

for some real β, then we say that $q(z) \in \mathcal{Q}(\beta)$, where

$$\beta := \begin{cases} \dfrac{4\alpha - 1}{2} & \left(\dfrac{1}{4} \leqq \alpha < \dfrac{1}{2}\right) \\[2ex] \dfrac{3 - 4\alpha}{2} & \left(\dfrac{1}{2} \leqq \alpha \leqq \dfrac{3}{4}\right). \end{cases} \tag{1.4}$$

We note that the function $q(z)$ given by (1.2) is the extremal function for the class $\mathcal{Q}(\beta)$.

Example. If we take $\alpha = \dfrac{1}{2}$ in (1.2), then we readily see that

$$q(0) = 1 \text{ and } q(z) = \frac{1}{1 - \sqrt[3]{z^2}} = 1 + \sum_{n=1}^{\infty} z^{\frac{2n}{3}} \ (z \in \mathbb{U}).$$

Thus, for $z = re^{i\theta}$ $(0 < r < 1;\ 0 \leqq \theta < 2\pi)$, we have

$$\begin{aligned} \Re\big(q(z)\big) &= \frac{1}{2} + \frac{1 - r^{\frac{4}{3}}}{2\left(1 - 2r^{\frac{2}{3}} \cos\left(\frac{2}{3}\theta\right) + r^{\frac{4}{3}}\right)} \\ &\geqq \frac{1}{2} + \frac{1 - r^{\frac{2}{3}}}{2(1 + r^{\frac{2}{3}})} \\ &> \frac{1}{2}. \end{aligned} \tag{1.5}$$

Equation (1.5) shows us that $q(z) \in \mathcal{Q}\left(\frac{1}{2}\right)$.

2. A Set of Coefficient Inequalities

First of all, we investigate some coefficient inequalities for the function $q(z)$ concerning with our new class $\mathcal{Q}(\beta)$.

Theorem 1. *If a function $q(z)$ given by* (1.3) *satisfies the following inequality:*

$$\sum_{n=1}^{\infty} \left(\left| a_{\frac{2n}{3}} \right| + \left| a_{\frac{2n+1}{3}} \right| \right) \leqq \begin{cases} \dfrac{3 - 4\alpha}{2} & \left(\dfrac{1}{4} \leqq \alpha < \dfrac{1}{2}\right) \\[2ex] \dfrac{4\alpha - 1}{2} & \left(\dfrac{1}{2} \leqq \alpha \leqq \dfrac{3}{4}\right), \end{cases} \tag{2.1}$$

then $q(z) \in \mathcal{Q}(\beta)$.

Proof. By the definition for the function class $\mathcal{Q}(\beta)$, if $q(z)$ satisfies the following inequality:

$$|q(z) - 1| < 1 - \beta \qquad (z \in \mathbb{U}),$$

then we say that $q(z) \in \mathcal{Q}(\beta)$, where β is given by (1.4). We note for $z \in \mathbb{U}$ that

$$\begin{aligned} |q(z) - 1| &= \left| \sum_{n=1}^{\infty} \left(a_{\frac{2n}{3}} + a_{\frac{2n+1}{3}} z^{\frac{1}{3}} \right) z^{\frac{2n}{3}} \right| \\ &< \sum_{n=1}^{\infty} \left(\left| a_{\frac{2n}{3}} \right| + \left| a_{\frac{2n+1}{3}} \right| \right) \\ &\leqq \beta = \begin{cases} \dfrac{3 - 4\alpha}{2} & \left(\dfrac{1}{4} \leqq \alpha < \dfrac{1}{2}\right) \\[2ex] \dfrac{4\alpha - 1}{2} & \left(\dfrac{1}{2} \leqq \alpha \leqq \dfrac{3}{4}\right). \end{cases} \end{aligned}$$

Therefore, since (by hypothesis) $q(z)$ satisfies the coefficient inequality (2.1), we conclude that $q(z) \in \mathcal{Q}(\beta)$. $\square$

Letting $\alpha = \dfrac{1}{4}$ or $\alpha = \dfrac{3}{4}$ in Theorem 1, we deduce Corollary 1 below.

Corollary 1. *If a function $q(z)$ given by* (1.3) *satisfies the following inequality:*

$$\sum_{n=1}^{\infty} \left(\left| a_{\frac{2n}{3}} \right| + \left| a_{\frac{2n+1}{3}} \right| \right) \leqq 1,$$

then $q(z) \in \mathcal{Q}(0)$.

If we set $\alpha = \dfrac{1}{2}$ in Theorem 1, then we have the following corollary.

Corollary 2. *If a function $q(z)$ given by* (1.3) *satisfies the following inequality:*

$$\sum_{n=1}^{\infty} \left(\left| a_{\frac{2n}{3}} \right| + \left| a_{\frac{2n+1}{3}} \right| \right) \leqq \frac{1}{2},$$

then $q(z) \in \mathcal{Q}(\frac{1}{2})$.

Next, we consider a function $q(z)$ given by (1.3) with

$$a_{\frac{2n}{3}} = \left| a_{\frac{2n}{3}} \right| e^{i(\pi - \frac{2n}{3}\theta)} \tag{2.2}$$

and

$$a_{\frac{2n+1}{3}} = \left| a_{\frac{2n+1}{3}} \right| e^{i(\pi - \frac{2n+1}{3}\theta)} \tag{2.3}$$

for some θ $(0 \leqq \theta < 2\pi)$ and for $n = 1, 2, 3, \cdots$.

Theorem 2. *Let a function $q(z)$ be given by* (1.3) *with* (2.2) *and* (2.3). *Then $q(z)$ belongs to the class $\mathcal{Q}(\beta)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\left| a_{\frac{2n}{3}} \right| + \left| a_{\frac{2n+1}{3}} \right| \right) \leqq 1 - \beta, \tag{2.4}$$

where β is defined by (1.4).

Proof. First of all, by appealing to Theorem 1, we know that $q(z)$ belongs to the function class $\mathcal{Q}(\beta)$ if $q(z)$ satisfies the coefficient inequality (2.4). We next suppose that $q(z) \in \mathcal{Q}(\beta)$. Then, upon letting $z = re^{i\theta}$ $(0 < r < 1)$, we see that

$$
\begin{aligned}
(2.5) \qquad & \Re(q(z)) \\
&= 1 + \Re\left(\sum_{n=1}^{\infty} \left(a_{\frac{2n}{3}} + \left(a_{\frac{2n+1}{3}} z^{\frac{1}{3}} \right) z^{\frac{2n}{3}} \right) \right) \\
&= 1 + \Re\left(\sum_{n=1}^{\infty} \left(\left| a_{\frac{2n}{3}} \right| + \left| a_{\frac{2n+1}{3}} \right| r^{\frac{1}{3}} \right) r^{\frac{2n}{3}} e^{i\pi} \right) \\
&= 1 - \sum_{n=1}^{\infty} \left(\left| a_{\frac{2n}{3}} \right| + \left| a_{\frac{2n+1}{3}} \right| r^{\frac{1}{3}} \right) r^{\frac{2n}{3}} \\
&> \beta.
\end{aligned}
$$

This last inequality in (2.5) shows us that the inequality (2.4) holds true for $r \to 1-$. This completes the proof of Theorem 2. □

Taking $\beta = 0$ in Theorem 2, we have Corollary 3 below.

Corollary 3. *Let $q(z)$ be given by* (1.3) *with* (2.2) *and* (2.3). *Then $q(z) \in \mathcal{Q}(0)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\left| a_{\frac{2n}{3}} \right| + \left| a_{\frac{2n+1}{3}} \right| \right) \leq 1.$$

Furthermore, if we set $\beta = \dfrac{1}{2}$ in Theorem 2, then we are led easily to the following corollary.

Corollary 4. *Let $q(z)$ be given by* (1.3) *with* (2.2) *and* (2.3). *Then $q(z) \in \mathcal{Q}(\frac{1}{2})$ if and only if*

$$\sum_{n=1}^{\infty} \left(\left| a_{\frac{2n}{3}} \right| + \left| a_{\frac{2n+1}{3}} \right| \right) \leq \frac{1}{2}.$$

Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
and
China Medical University
Taichung 40402, Taiwan
Republic of China
E-mail address: harimsri@math.uvic.ca

Department of Mathematics
Faculty of Education
Yamato University
Katayama 2-5-1, Suita Osaka 564-0082
Japan
E-mail address: owa.shigeyoshi@yamato-u.ac.jp

3. An Open Problem

Since we have not yet found any extremal functions for Theorem 1 *and* Theorem 2, we find it to be worthwhile to pose the following *open* problem arising from our present investigation.

Open Problem. What kind of functions $q(z)$ can be the *extremal* functions for Theorem 1 and Theorem 2?

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