J COMP SCI APPL MATH
Journal of Computer Science and Applied Mathematics **2** (2016), no.1, 15-17
ISSN:1857-9582 https://doi.org/10.37418/jcsam.2.1.3

SOME RESULTS ABOUT A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. In this article the author continues the examination of the class $\widetilde{K}$, which is a subclass of the class of close-to-convex functions. In addition to the already obtained sharp growth and distortion results for this class there is given the radius of convexity of the class $\widetilde{K}$ and also a result concerning the derivatives of the functions from the class $\widetilde{K}$.

1. Introduction and definitions

Let us first recall the main necessary definitions. Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic and univalent in the open unit disk $E = \{z : |z| < 1\}$.

Let C denote the class of convex functions [1]:

$$f(z) \in C \text{ if and only if for } z \in E,$$
$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0.$$

By S^* we denote the class of starlike functions [2]:

$$f(z) \in S^* \text{ if and only if for } z \in E, \Re\frac{zf'(z)}{f(z)} > 0.$$

A function $f(z)$ analytic in E is said to be close-to-convex in E, if there exists a function $g(z) \in S^*$ such that for $z \in E$

$$\Re\frac{zf'(z)}{g(z)} > 0.$$

The class of such functions is denoted by K, [4].

The classes S, K, S^* and C are related by the proper inclusions

$$C \subset S^* \subset K \subset S.$$

Now we will consider a class $\widetilde{K}$ defined as follows: Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E. Then $f(z) \in \widetilde{K}$ if and only if there exists a function $g(z) \in C$ such that for $z \in E$

$$\Re\frac{zf'(z)}{g(z)} > 0. \tag{1.1}$$

Since $C \subset S^*$, it follows that $\widetilde{K} \subset K$ and so, the functions in $\widetilde{K}$ are univalent.

Let P be the class of functions $h(z)$ given by $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, which are analytic and have positive real part in E.

Let Ω be the class of functions ω analytic in E such that $\omega(0) = 0$ and $|\omega(z)| \leq |z|$ for $z \in E$.

2. Known results

Theorem 2.1. ([3]) *If $g(z) \in C$, with $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then*

(i) $$|b_n| \leq 1 \qquad (n = 2, 3, \ldots),$$

(ii) $$\left|b_3 - \mu b_2^2\right| \leq \max\left(\frac{1}{3}, |1-\mu|\right),$$

(iii) $$\left|b_3 - \mu b_2^2\right| \leq \frac{1}{3}\left(1 - |b_2|^2\right).$$

Theorem 2.2. ([5]) *Let $h(z) \in P$, with $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then*

(i) $$|c_n| \leq 2 \qquad (n = 1, 2, \ldots),$$

(ii) $$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}.$$

Equality holds when $h(z) = \dfrac{1+z}{1-z}$.

2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. univalent functions, close-to-convex functions.

Theorem 2.3. ([3]) *Let $g(z) \in C$, with $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then, for $z = re^{i\theta} \in E$,*

$$\begin{aligned} \frac{r}{1+r} &\le |g(z)| \le \frac{r}{1-r}, \\ \frac{1}{(1+r)^2} &\le |g'(z)| \le \frac{1}{(1-r)^2}, \\ \frac{1}{1+r} &\le \left|\frac{zg'(z)}{g(z)}\right| \le \frac{1}{1-r}. \end{aligned}$$

Equality holds if and only if $g(z) = \dfrac{z}{(1-\varepsilon z)}$, $|\varepsilon| = 1$.

Theorem 2.4. ([2]) *A function $h(z) \in P$ if, and only if*

$$h(z) = \frac{1+\omega(z)}{1-\omega(z)}, \qquad z \in E,$$

where $\omega \in \Omega$.

Theorem 2.5. ([6]) *If $h(z) \in P$, then for $z = re^{i\theta} \in E$*

$$\begin{aligned} &(i) & \frac{1-r}{1+r} &\le |h(z)| \le \frac{1+r}{1-r}, \\ &(ii) & \left|\frac{zh'(z)}{h(z)}\right| &\le \frac{2r}{1-r^2}, \\ &(iii) & |h'(z)| &\le \frac{2\Re h(z)}{1-r^2}. \end{aligned}$$

Equality is attained when $h(z) = \dfrac{1+\varepsilon z}{1-\varepsilon z}$, $|\varepsilon| = 1$.

3. Some of the basic properties of functions in $\widetilde{K}$

Theorem 3.1. *Let $f(z) \in \widetilde{K}$. Then for $z = re^{i\theta} \in E$,*

$$\frac{1-r}{(1+r)^2} \le |f'(z)| \le \frac{1+r}{(1-r)^2},$$

$$-\ln(1+r) + \frac{2r}{1+r} \le |f(z)| \le \ln(1-r) + \frac{2r}{1-r}.$$

Each inequality is sharp for $f_0(z)$ defined by

$$f_0(z) = \overline{x}\log(1-xz) + \frac{zx}{1-xz}, \quad \text{with } |x| = 1. \tag{3.1}$$

Theorem 3.2. *Let $f(z) \in \widetilde{K}$, with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then for $z \in E$,*

$$|a_n| \le 2 - \frac{1}{n} \quad \text{for} \quad n \ge 2.$$

Equality is attained for $f_0(z)$ defined in (3.1).

Theorem 3.3. *Let $f(z) \in \widetilde{K}$ and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{5}{3} - \dfrac{9}{4}\mu, & \text{if } \mu \le \dfrac{2}{9}, \\ \dfrac{2}{3} + \dfrac{1}{9\mu}, & \text{if } \dfrac{2}{9} \le \mu \le \dfrac{2}{3}, \\ \dfrac{5}{6}, & \text{if } \dfrac{2}{3} \le \mu \le 1. \end{cases}$$

For each μ, there is a function in $\widetilde{K}$ such that equality holds.

These results have been obtained by the author [7] by means of classic methods by using the already known results mentioned in Section 2.

Other kind of author's results on some classes of functions related to the class of the convex functions can be found in the recent papers [8] and [9].

4. Some additional results for the class $\widetilde{K}$

Theorem 4.1. *If $f(z) \in \widetilde{K}$, then, for $z = re^{i\theta} \in E$*

$$|\arg f'(z)| \le \arcsin\frac{2r}{1+r^2} + \arcsin r.$$

The bound is sharp.

Proof. Since $f(z) \in \widetilde{K}$, we may write from Theorem 2.4

$$\frac{zf'(z)}{g(z)} = \frac{1+w(z)}{1-w(z)},$$

where $w(z)$ is analytic in E with $w(0) = 0$ and $|w(z)| \le 1$. Also, it is known [2] that the image of the closed disc $|z| \le r$ under the transformation $h(z) = \dfrac{1+w(z)}{1-w(z)}$ is contained in the closed disc with the center A and radius ρ, where

$$A = \frac{1+r^2}{1-r^2}, \quad \rho = \frac{2r}{1-r^2}.$$

Thus we have

$$\left|\frac{zf'(z)}{g(z)} - \frac{1+r^2}{1-r^2}\right| \le \frac{2r}{1-r^2}, \quad g \in C.$$

This implies that

$$\left|\arg\frac{zf'(z)}{g(z)}\right| \le \arcsin\frac{2r}{1+r^2},$$

or

$$|\arg f'(z)| \le \arcsin\frac{2r}{1+r^2} + \left|\arg\frac{g(z)}{z}\right|.$$

For the function $g(z) \in C$,

$$\left|\arg\frac{g(z)}{z}\right| \le \arcsin r \qquad (\textit{see}[1])$$

and so $|\arg f'(z)| \le \arcsin\dfrac{2r}{1+r^2} + \arcsin r$ as required.

To show that the inequality is sharp, chose θ_1 and θ_2 so that

$$\frac{zf'(z)}{g(z)} = \frac{1+\theta_1 z}{1-\theta_1 z}, \quad |\theta_1| = 1 \tag{4.1}$$

and

$$g(z) = \frac{z}{1+\theta_2 z}, \quad |\theta_2| = 1, \tag{4.2}$$

where $\theta_1 = \dfrac{ir}{z}$ at any point on $|z| = r$ so that

$$\arg\frac{zf'(z)}{g(z)} = \arcsin\frac{2r}{1+r^2}. \tag{4.3}$$

Taking $\theta_2 = \dfrac{r}{z}\left[-r + i\sqrt{1-r^2}\right]$ at any point on $|z| = r$ in (4.2), we obtain

$$\arg\frac{g(z)}{z} = \arg(1+\theta_2 z) = \arcsin r. \tag{4.4}$$

Combining (4.3) and (4.4) shows that for this choice of $f(z)$ and $g(z)$ and θ_1 and θ_2

$$\arg f'(z) = \arcsin\frac{2r}{1+r^2} + \arcsin r$$

on $|z| = r$. □

Theorem 4.2. *If $f(z) \in \widetilde{K}$, then $f(z)$ maps $\left\{z : |z| < \dfrac{1}{3}\right\}$ onto a convex set. The function $f_0(z)$ given by (3.1) shows that this result is the best possible.*

Proof. It follows from (1.1) that we can write $zf'(z) = g(z)h(z)$ for $g(z) \in C$ and $h(z) \in P$. Differentiating logarithmically, we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)}$$

from which it follows that

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \Re\frac{zg'(z)}{g(z)} - \left|\frac{zh'(z)}{h(z)}\right| \tag{4.5}$$

since $g(z) \in C$, we know from Theorem 2.3 that

$$\Re\frac{zg'(z)}{g(z)} \geq \frac{1}{1+r}. \tag{4.6}$$

Also from Theorem 2.5,

$$\left|\frac{zh'(z)}{h(z)}\right| \leq \frac{r}{1-r^2}. \tag{4.7}$$

Thus from (4.5), (4.6) and (4.7) we obtain

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \frac{1}{1+r} - \frac{2r}{1-r^2} = \frac{1-3r}{1-r^2}.$$

Hence,

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad \text{for} \quad |z| < \frac{1}{3},$$

and the result is proved. □

Acknowledgements

This article is supported financially by the bilateral project "Analysis, Geometry and Topology" between Bulgarian Academy of Sciences (BAN) and Macedonian Academy of Science and Arts (MANU).

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