A MIXED FORMULATION FOR A BENDING DOMINATED KOITER SHELL WITH OBSTACLE

Hanen Ferchichi

ABSTRACT. In this paper, we present a mixed formulation for a bending dominated Koiter shell with obstacle in order to avoid numerical locking or the deterioration of the convergence when the small parameter the thickness goes to zero. This formulation is a combination between the free locking mixed formulation presented in [1, 10] and the Koiter's model with obstacle for flexural shell proposed in [6].

1. THE KOITER MODEL FOR A BENDING DOMINATED SHELL

Greek indices take their values in the set $\{1, 2\}$ and the Latin indices take their values in $\{1, 2, 3\}$. Products containing repeated indices are summed. Let ω be an open convex domain of $\mathbb{R}^2$. We consider a shell whose midsurface is given by $S = \varphi(\bar{\omega})$ where $\varphi \in C^3(\omega, R^3)$ is an injective mapping. Let $\vec{a}_\alpha = \varphi_{,\alpha}$, $\alpha = 1, 2$; $\vec{a}_3 = \frac{\vec{a}_1 \wedge \vec{a}_2}{\|\vec{a}_1 \wedge \vec{a}_2\|}$ be the covariant basis vectors and $\vec{a}^\alpha$ defined by $\vec{a}^\alpha \cdot \vec{a}_\alpha = \delta^\alpha_\beta$; $\vec{a}_3 = \vec{a}^3$ be the contravariant basis vectors. Let ε be the shell thickness. The first and second fundamental forms of the midsurface are defined componentwise by

$$a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta, \qquad b_{\alpha\beta} = \vec{a}_3 \cdot \vec{a}_{\alpha,\beta} = -\vec{a}_\alpha \cdot \vec{a}_{3,\beta}.$$

Let $a = \|\vec{a}_1 \wedge \vec{a}_2\|^2$ be the determinant of $(a_{\alpha\beta})_{\alpha\beta}$. We note $a^{\alpha\beta} = \vec{a}^\alpha \cdot \vec{a}^\beta$ the first fundamental form contravariant components and $b^\alpha_\gamma = a^{\alpha\beta} b_{\beta\gamma}$ the mixed

---

[1]*corresponding author*

Key words and phrases. Koiter shell, bending dominated, mixed formulation, obstacle problem.

components of the second fundamental form. For a displacement field $\vec{u}$, the linearized changes of the curvature tensor $\underline{\underline{\Upsilon}} = (\Upsilon_{\alpha\beta})_{\alpha,\beta}$ and of the membrane tensor $\underline{\underline{\Lambda}} = (\Lambda_{\alpha\beta})_{\alpha,\beta}$ read in covariant components [2,3]:

(1.1)
$$\Upsilon_{\alpha\beta}(\vec{u}) = u_{3/\alpha\beta} - b_\alpha^\sigma b_{\sigma\beta} u_3 + b_\alpha^\sigma u_{\sigma/\beta} + b_\beta^\sigma u_{\sigma/\alpha} + b_{\beta/\alpha}^\sigma u_\sigma,$$
$$\Lambda_{\alpha\beta}(\vec{u}) = \tfrac{1}{2}(u_{\alpha/\beta} + u_{\beta/\alpha}) - b_{\alpha\beta} u_3,$$

where the covariant derivative of displacement components and second fundamental form mixed components is given by:

$$u_{\alpha/\beta} = u_{\alpha,\beta} - \Gamma_{\alpha\beta}^\delta u_\delta, \quad u_{3/\alpha\beta} = u_{3,\alpha\beta} - \Gamma_{\alpha\beta}^\delta u_{3,\delta},$$
$$b_{\alpha/\rho}^\beta = b_{\alpha,\rho}^\beta + \Gamma_{\rho\sigma}^\beta b_\alpha^\sigma - \Gamma_{\alpha\rho}^\sigma b_\sigma^\beta, \quad \Gamma_{\alpha\beta}^\delta = a^\delta \cdot a_{\alpha\beta}.$$

Let $\underline{\underline{\underline{E}}} = (E^{\alpha\beta\lambda\mu})_{\alpha\beta\lambda\mu}$ be the elasticity tensor, assumed to be elliptic as well as its inverse, given by $E^{\alpha\beta\lambda\mu} = \frac{\epsilon}{2(1+\nu)}(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu})$, where $\epsilon > 0$ and $\nu \in (0, \frac{1}{2})$ are respectively the Young's modulus and Poisson ratio. We suppose the shell clamped on a part $\Gamma \neq \emptyset$ of its boundary and set

$$V = \{\vec{v} = v_i a^i, v_\alpha \in H^1(\omega), v_3 \in H^2(\omega); v_i = \frac{\partial v_3}{\partial n} = 0 \text{ on } \Gamma\}.$$

Note that V is a Hilbert space when endowed with the norm

$$||\vec{v}||_V = (\sum_\alpha ||v_\alpha||_{H^1}^2 + ||v_3||_{H^2}^2)^{1/2}.$$

Consider the bending-dominated Koiter shell problem

(1.2)   (**P**) $\begin{cases} \text{Find } \vec{u} \in V \\ \frac{\varepsilon^3}{12}\int_\omega E^{\alpha\sigma\lambda\mu}\Upsilon_{\alpha\sigma}(\vec{u})\Upsilon_{\lambda\mu}(\vec{v})\sqrt{a}dx + \varepsilon \int_\omega E^{\alpha\sigma\lambda\mu}\Lambda_{\alpha\sigma}(\vec{u})\Lambda_{\lambda\mu}(\vec{v})\sqrt{a}dx \\ \quad = \int_\omega \vec{F} \cdot \vec{v}\sqrt{a}dx. \ \forall \vec{v} \in V. \end{cases}$

The asymptotic analysis, as the shell thickness goes to zero, show different behaviors of the linear elastic shell depending on the boundary conditions, the geometry of the middle surface and the applied force. The zero of the membrane energy subspace displacements has a crucial role to distinguish this behavior. When this space is different to zero, the shell is in bending dominated state. In this case and in order to obtain a well posed and non trivial limit problem, we scale the external forces in the form [1], [4], [9]: $\vec{F} = \varepsilon^3 \vec{f}$, where $\vec{f}$ is independent of ε.

$\tilde{A}$ is a bilinear form corresponding to internal energy given by

$$\tilde{A}(\vec{u}; \vec{v}) = \frac{\varepsilon^3}{12} \int_\omega E^{\alpha\sigma\lambda\mu} \Upsilon_{\alpha\sigma}(\vec{u}) \Upsilon_{\lambda\mu}(\vec{v}) \sqrt{a} dx + \varepsilon \int_\omega E^{\alpha\sigma\lambda\mu} \Lambda_{\alpha\sigma}(\vec{u}) \Lambda_{\lambda\mu}(\vec{v}) \sqrt{a} dx.$$

$\tilde{L}$ is a linear form corresponding to external forces

$$(1.3) \qquad\qquad \tilde{L}(\vec{v}) = \varepsilon^3 \int_\omega \vec{f} \cdot \vec{v} \sqrt{a} dx.$$

Note that $\tilde{A}$ is continuous and coercive on V and $\tilde{L}$ is continuous. Then, the problem has a unique solution [2, 3]. Equivalently, $\vec{u}$ is determined as the minimizer over V of the energy functional

$$\begin{aligned}
\mathbb{E}(\vec{v}) &= \mathbb{E}^B + \mathbb{E}^M - \tilde{L} = \frac{\varepsilon^3}{2} \int_\omega \frac{a^{\alpha\beta\gamma\delta}}{12} \Upsilon_{\alpha\beta}(\vec{v}) \Upsilon_{\gamma\delta}(\vec{v}) \sqrt{a} dx \\
(1.4) \qquad &\quad + \frac{\varepsilon}{2} \int_\omega a^{\alpha\beta\gamma\delta} \Lambda_{\alpha\beta}(\vec{v}) \Lambda_{\gamma\delta}(\vec{v}) \sqrt{a} dx - \tilde{L}(\vec{v}).
\end{aligned}$$

2. THE KOITER MODEL FOR A BENDING DOMINATED SHELL WITH OBSTACLE

Assuming that the deformed middle surface of the shell remains in a given half space $\mathbb{H} = \{y \in \mathbb{R}^3; oy.p \geq 0\}$, where p is a given non-zero vector in $\mathbb{R}^3$ [6]. Then, the unknown displacement field $u_i(x)a^i(x)$ of the middle surface $S = \varphi(\overline{\omega})$ is determined such that the energy $\mathbb{E}(\vec{v})$ is minimized over a strict subset U of V given by:

$$(2.1) \qquad U = \{\vec{u} \in V; (\varphi(x) + u_i(x)a^i(x)).p \geq 0 \; \forall x \in \omega\}.$$

The constrained minimization problem will be the following:

$$(2.2) \qquad (\mathbf{P_{min}}) \begin{cases} \text{Find } \vec{u} \in U, \\ \mathbb{E}(\vec{u}) = \inf_{\vec{v} \in U} \mathbb{E}(\vec{v}). \end{cases}$$

Since U is a non-empty closed and convex subset of V, the problem (P_{min}) has a unique solution and is equivalent to the problem $(\mathbf{P_U})$ of variational inequalities [6]:

$$(\mathbf{P_U}) \begin{cases} \text{Find } \vec{u} \in U \\ \frac{1}{12} \int_\omega E^{\alpha\sigma\lambda\mu} \Upsilon_{\alpha\sigma}(\vec{u}) \Upsilon_{\lambda\mu}(\vec{v} - \vec{u}) \sqrt{a} dx + \varepsilon^{-2} \int_\omega E^{\alpha\sigma\lambda\mu} \Lambda_{\alpha\sigma}(\vec{u}) \Lambda_{\lambda\mu}(\vec{v} - \vec{u}) \sqrt{a} dx \\ \qquad \geq \int_\omega \vec{f}.(\vec{v} - \vec{u}) \sqrt{a} dx, \; \forall \vec{v} \in U, \end{cases}$$

The asymptotic behavior of the constrained problem (P_U) and the unconstrained problem (P) are similar in the sensa that the solutions of their limit problems are pure bending displacements. In fact, the limit problem of (P) is defined on the space of admissible pure bending displacements:

$$V_F = \{\vec{v} = v_i a^i, v_\alpha \in H^1(\omega), v_3 \in H^2(\omega); v_i = \frac{\partial v_3}{\partial n} = 0 \text{ on } \Gamma \text{ and } \Lambda_{\alpha\sigma}(\vec{v}) = 0\}.$$

The solution of the limit problem of (P_U) is defined on U_F the non-empty closed and convex subspace of V_F such that the deformed middle surface remains in $\mathbb{H}$ [6]:

$$U_F = \{\vec{v} \in V_F; (\varphi(x) + v_i(x)a^i(x)) \cdot p \geq 0 \ \forall x \in \omega\},$$

and satisfies the inequality [6]:

$$\frac{1}{12} \int_\omega E^{\alpha\sigma\lambda\mu} \Upsilon_{\alpha\sigma}(\vec{u}) \Upsilon_{\lambda\mu}(\vec{v} - \vec{u}) \sqrt{a} dx \geq \int_\omega \vec{f} \cdot (\vec{v} - \vec{u}) \sqrt{a} dx, \ \ \forall \, \vec{v} \in U_F.$$

3. A MIXED FORMULATION FOR A BENDING DOMINATED KOITER SHELL WITH OBSTACLE

It is shown in literature that the deterioration of the approximation occurs for small thickness when using the standard finite element methods for a bending dominated shell [1, 4, 9].

This phenomenon is called the numerical or the membrane locking. The authors in [1] provide a free locking mixed formulation. The same technique is used in [10] to provide a stable numerical solution for the Koiter shell model. The numerical studies of the mixed numerical schemes proposed show a good properties of convergence as expected [5, 7, 8].

The numerical trouble is in relation with the subspace of the limit problem. The bending dominated behavior is showed for bending dominated Koiter shell with obstacle [6] therefore membrane locking is expected for numerical solution.

To apply the remedy of a mixed solution, we introduce a new variable $\underline{\underline{\lambda}}$ which represents the membrane stress aside a multiplicator factor. We set, for a real c_0 such that $0 < c_0 < \varepsilon^{-2}$,

$$\underline{\underline{\lambda}} = (\lambda_{\alpha\gamma})_{\alpha\gamma}, \ \ \lambda^{\alpha\gamma} = (\frac{1}{\varepsilon^2} - c_o) E^{\alpha\gamma\sigma\mu} \Lambda_{\sigma\mu}(\vec{u}).$$

Let the set:

$$W = \{\underline{\underline{\chi}}; \chi^{\alpha\beta} \in L^2(\omega)\}.$$

We define the bilinear form A, B and C by:

$$A(\vec{u}; \vec{v}) = \int_\omega \frac{1}{12} E^{\alpha\sigma\lambda\mu}\Upsilon_{\alpha\sigma}(\vec{u})\Upsilon_{\lambda\mu}(\vec{v})\sqrt{a}dx + c_0 \int_\omega E^{\alpha\sigma\lambda\mu}\Lambda_{\alpha\sigma}(\vec{u})\Lambda_{\lambda\mu}(\vec{v})\sqrt{a}dx,$$

$$B(\vec{v}; \underline{\underline{\xi}}) = \int_\omega \Lambda_{\alpha\sigma}(\vec{v})\xi^{\alpha\sigma}\sqrt{a}dx, \ C(\underline{\underline{\lambda}};$$

$$\underline{\underline{\xi}}) = \int_\omega (E^{-1})_{\alpha\sigma\delta\mu}\lambda^{\delta\mu}\xi^{\alpha\sigma}\sqrt{a}dx.$$

We define the bilinear form:

$$\widetilde{\widetilde{A}}(\vec{u}, \underline{\underline{\lambda}}; \vec{v}, \hat{\underline{\underline{\lambda}}}) = A(\vec{u}; \vec{v}) + B(\vec{v}, \underline{\underline{\lambda}}) - B(\vec{u}, \hat{\underline{\underline{\lambda}}}) + \frac{t^2}{1 - c_o t^2}C(\underline{\underline{\lambda}}, \hat{\underline{\underline{\lambda}}})$$

Then the total energy is given by:

$$\mathbb{E}(\vec{u}, \underline{\underline{\lambda}}) = \mathbb{E}^B + \mathbb{E}^M - \tilde{L} = \frac{1}{2}\{\varepsilon^3 \widetilde{\widetilde{A}}(\vec{u}, \underline{\underline{\lambda}}, \vec{u}, \underline{\underline{\lambda}})\} - \tilde{L}(\vec{u}) \text{ for } (\vec{u}, \underline{\underline{\lambda}}) \in U \times W,$$

where $\tilde{L}(\vec{u})$ is given by (1.3) and

$$\mathbb{E}^B = \frac{\varepsilon^3}{2}\int_\omega \frac{1}{12}E^{\alpha\sigma\lambda\mu}\Upsilon_{\alpha\sigma}(\vec{u})\Upsilon_{\lambda\mu}(\vec{u})\sqrt{a}dx,$$

$$\mathbb{E}^M = \frac{\varepsilon^3}{2}[c_0 \int_\omega E^{\alpha\sigma\lambda\mu}\Lambda_{\alpha\sigma}(\vec{u})\Lambda_{\lambda\mu}(\vec{v})\sqrt{a}dx + \frac{\varepsilon^2}{1 - c_o\varepsilon^2}\int_\omega (E^{-1})_{\alpha\sigma\delta\mu}\lambda^{\delta\mu}\lambda^{\alpha\sigma}\sqrt{a}dx].$$

We endow W by the standard L^2 product norm and by the semi norm:

$$|||\underline{\underline{\lambda}}||| = \sup_{\vec{v}\in V} \frac{B(\vec{v};\underline{\underline{\lambda}})}{\|\vec{v}\|}.$$

Theorem 3.1. *The minimisation problem*

$$(\mathbf{M_{min}}) \begin{cases} \text{Find } (\vec{u}, \underline{\underline{\lambda}}) \in U \times W \text{ such that} \\ \mathbb{E}(\vec{u}, \underline{\underline{\lambda}}) = \inf_{(\vec{v}, \hat{\underline{\underline{\lambda}}})\in U\times W} \mathbb{E}(\vec{v}, \hat{\underline{\underline{\lambda}}}) \end{cases}$$

has a unique solution and is equivalently to the problem $(\mathbf{M_U})$:

(3.1) $$(\mathbf{M_U}) \begin{cases} \text{seek pairs } (\vec{u}, \underline{\underline{\lambda}}) \in U \times W \text{ such that} \\ A(\vec{u}; \vec{v} - \vec{u}) + B(\vec{v}; \underline{\underline{\lambda}}) \geq l(\vec{v} - \vec{u}) \ \forall \ \vec{v} \in U, \\ B(\vec{u}; \hat{\underline{\underline{\lambda}}}) - \frac{\varepsilon^2}{1 - c_o t^2}C(\underline{\underline{\lambda}}; \hat{\underline{\underline{\lambda}}}) = 0 \ \forall \ \hat{\underline{\underline{\lambda}}} \in W, \end{cases}$$

where $l(\vec{v}) = \int_\omega \vec{f} \cdot \vec{v}\sqrt{a}dx.$

Proof. The bilinear forms A, B, C are continuous respectively on $V \times V$, $V \times W$ and $W \times W$. We note also that A is $V-$elliptic and C is $W-$elliptic, so the bilinear form $\widetilde{A}$ is $V \times W$ elliptic. Moreover, $U \times W$ is a non empty closed and convex subspace of $V \times W$. Then the problem (M_{min}) has a unique solution. $\square$

Remark 3.1. *The problem* (3.1) *correspond to the saddle point problem*

$$(3.2) \qquad \inf_{\vec{u} \in U} \sup_{\underline{\underline{\lambda}} \in W} \{ \tfrac{1}{2} A(\vec{u}; \vec{u}) + B(\vec{u}; \underline{\underline{\lambda}}) - \tfrac{\varepsilon^2}{2(1-c_o\varepsilon^2)} C(\underline{\underline{\lambda}}; \underline{\underline{\lambda}}) - l(\vec{u}) \}.$$

It has a unique solution. The primal variable is the solution of the initial problem (2.2).

REFERENCES

[1] D.N. ARNOLD, F. BREZZI: *Locking-free finite element methods for shells*, Mathematics of Computation, **66** (1997), 1–14.

[2] M. BERNADOU: *Méthodes d'éléments finis pour les problèmes de coques minces*, in: RMA, Vol. 33, Masson, Paris, 1994.

[3] M. BERNADOU, P.G. CIARLET: *Sur l'éllipticité du modèle linéaire de coques de Koiter*, Lecture Notes in Economics and Mathematical systems, Springer Verlag, Berlin, **134** (1976), 89–136.

[4] D. CHAPELLE, K.J. BATHE: *The finite element analysis of shells-fundamentals*, Springer-Verlag, 2003.

[5] C. CHINOSI, G. SACCHI: *Partial selective reduced integration for the approximation of bending dominated shells*, Fourth International Colloquim on Computation of shell and Spatial Structures, June 5-7, Chania-Crete, Greece, 2000.

[6] P.G. CIARLET, P. PIERSANTI: *A confinement problem for a linearly elastic Koiters shell*, C.R.Acad. Sci. Paris, Ser.1 **357** (2019), 221–230.

[7] H. FERCHICHI, S.M. AOUADI: *Stabilité numérique d'une formulation mixte pour les coques*, C. R. Acad. Sci. Paris, Ser. I, **346** (2008), 1103–1108.

[8] H. FERCHICHI, S.M. AOUADI: *A conforming locking-free approxiamtion for a Koiter shell*, Applied Mathematics and Computation, **339** (2018), 374–389.

[9] J.SANCHEZ-HUBERT, E.SANCHEZ-PALENCIA: *Coques élastiques minces, propriétés asymptotiques*, Masson, Paris, 1997.

[10] T. SUN: *A locking free mixed finite element method for the Koiter shell model*, Iterative Methods in Scientific Computation, J. Wang, M. B. Allen, B. Chen, T. Mathew (eds.), 1998, 1–6.

Department of Mathematics, Faculty of sciences of Sakaka

JOUF University

Sakaka, Aljouf

Email address: hferchichi@ju.edu.sa