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L-MMH: AN IMPROVED AND NOVEL MODEL FOR GROWTH

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ABSTRACT. Proposing a function for modelling growth is an important development for the curve fitting of data. This study gives a derivation for a new mathematical equation for growth and reports some significant features of this model.

1. INTRODUCTION

Studies dealing with the growth models found a wide area for itself in literature. For example, Gangying and Weitang (1996), defined the Sloboda growth model and they implemented its application on Chinese fir plantation and they stated that this model has satisfactory application effects. Tewari et. al. (2007) evaluated and compared some existing height growth equations for "Acacia nilotica and Eucalyptus" species by using Chapman-Richards generalization of Bertalanffy, Korf and Sloboda models in their study. Another important mortality function is Gompertz (1825). There are lots of studies about properties of Gompertz and applications of this model. Lenart (2014) defined the moments of the Gompertz distribution taking advantage of Laplace transform. Estimation of parameters by means of least squares estimation and least squares fitting (Wu

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et. al., 2004; Jukić et. al., 2004) and by means of maximum likelihood estimation (Garg et al.,1970) have been found. Bolton et al. (2015) presented a new model and applied it to the real data set for tumor growth. Shabanisamghabady and Tanaka (2016) compared five growth models (Exponential, Gompertz, Logistic, Generalized Logistic, von Bertalanffy) in the prediction of glioblastoma growth and determined the optimal model with the lsqcurvefit function.

Other important growth functions are proposed by Michaelis and Menten (1913) and Hill (1910, 1913) in the same period. Michaelis and Menten's (1913) equation has a hyperbolic form of $y(t) = \frac{y_\infty \mathrm{X}}{k+X}$ where Y is reaction speed in an enzyme and any parameter k>0. Hill's (1910, 1913) equation is $y(t) = \frac{y_\infty X^n}{k+X^n}$ having sigmoidal form for similar assumptions. But these models were not used directly for modelling the growth. Instead, they created an important infrastructure for future growth models. For example, Mercer et al. (1978) proposed an equation to model the biological efficiency of alternate nutrient sources inspiring the Michaelis and Menten (1913) and Hill (1910, 1913) equations. Also, Morgan et al. (1975) applied a general form of Michaelis and Menten (1913) and Hill (1910, 1913) type functions to evaluate nutritional requirements of nutrient sources.

In the first part of the study, another important Michaelis and Menten (1913) and Hill (1910, 1913) type function is given by using logarithmic function and called as L-MMH growth model. At that section construction of the theroetical formula is given matehmatically. In the second part some important properties of this growth model is obtained. Lastly, conclusion section is given.

2. Construction of L-MMH Growth Function

In this section a Michaelis and Menten (1913) and Hill (1910, 1913) type mathematical expression for the growth is obtained. Since it contains logarithmic function and has similar form as Michaelis and Menten (1913) and Hill (1910, 1913) models, this expression is called as L-MMH growth model

Growth Rate

Let us first define the L-MMH growth rate as follows:

$$\frac{\mathrm{dy}\,(t)}{\mathrm{dt}} = \frac{y_\infty - y\,(t)}{t\,(k+\ln t)}, \tag{2.1}$$

where $t > 0,\ y_\infty = \lim_{t\to\infty} y(t)$, k is any real number in $0 \le k \le 1$.

The Instant Rate:

Arranging equation (2.1) as proportion, the instant rate of change in $y\,(t)$ can be reached as follows:

$$\frac{\frac{\mathrm{dy}(t)}{\mathrm{dt}}}{y_\infty - y\,(t)} = \frac{1}{t\,(k+\ln t)}. \tag{2.2}$$

Current size of y at time t is shown by $y\,(t)$ and to find $y\,(t)$ we should take integration in equation (2.2):

$$\int_0^t \frac{\frac{\mathrm{dy}(t)}{\mathrm{dt}}}{y_\infty - y\,(t)}\,\mathrm{dt} = \int_{t_0}^t \frac{1}{t\,(k+\ln t)}\mathrm{dt} \tag{2.3}$$

$$-\ln\left(y_\infty - y\,(t)\right) + \ln\left(y_\infty - y\,(0)\right) = \ln\left(k+\ln t\right) - \ln\left(k + \ln t_0\right)$$

$$\ln\frac{y_\infty - y\,(0)}{y_\infty - y\,(t)} = \ln\frac{k+\ln t}{k+\ln t_0} \quad\Longleftrightarrow\quad \frac{y_\infty - y\,(0)}{y_\infty - y\,(t)} = \frac{k+\ln t}{k+\ln t_0}.$$

Then,

$$y\,(t) = \frac{y_\infty \ln\frac{t}{t_0} + y\,(0)\ \left(k+\ln t_0\right)}{k+\ln t}. \tag{2.4}$$

3. Properties of L-MMH Growth Function

In this section some characteristic properties of newly defined growth model are given.

Inflection point:

The point where the function changing direction is called an inflection point and can be obtained by reaching the zero point (and changing sign) for the second derivative.

The second derivative of $y(t)$ with respect to t should be 0. In other words, we have to compute

$$\frac{d^2y(t)}{dt^2} = 0$$

$$\frac{d^2y(t)}{dt^2} = -\frac{t^{-2}(y_\infty - y\,(t))\ (k+\ln t+1)}{\left(k+\ln t\right)^2} = 0 \quad\Longleftrightarrow\quad t_{\text{inf}} = e^{-(k+1)}$$

To check the point $t_{\text{inf}} = e^{-(k+1)}$ is an inflection point, we should see the behavior of the curve about this point. Since the following conditions

$$t_{\text{inf}} < e^{-(k+1)} \leftrightarrow \frac{d^2y(t)}{dt^2} < 0 \quad \text{and} \quad t_{\text{inf}} > e^{-(k+1)} \leftrightarrow \frac{d^2y(t)}{dt^2} > 0$$

are satisfied then it can be declared that $t_{\text{inf}} = e^{-(k+1)}$ is an inflection point. This knowledge enables to say the proposed curve has a sigmoidal structure.

Population size at inflection point:

Population size at inflection point can be given as,

$$y\left(t_{\text{inf}}\right) = y_{\infty}\left(k + \ln t_0 + 1\right) - y\left(0\right)\left(k + \ln t_0\right).$$

The location of the inflection point:

Then the location of the inflection point for the proposed growth is

$$(\ e^{-(k+1)},\ y_{\infty}\left(k + \ln t_0 + 1\right) - y(0)\ \left(k + \ln t_0\right)).$$

The maximum specific growth rate:

By arranging equation (2.1) at inflection point t_{inf}, the maximum specific growth rate can be reached.

So,

$$\begin{aligned} y'\left(t_{\text{inf}}\right) &= \frac{y_{\infty} - y\left(t_{\text{inf}}\right)}{e^{-(k+1)}\ (-1)} = \frac{y\left(t_{\text{inf}}\right) - y_{\infty}}{e^{-(k+1)}} \\ &= \frac{y_{\infty}\left(k + \ln t_0 + 1\right) - y\left(0\right)\left(k + \ln t_0\right) - y_{\infty}}{e^{-(k+1)}} = \frac{\left(k + \ln t_0\right)\ \left(y_{\infty} - y\left(0\right)\right)}{e^{-(k+1)}} \end{aligned}$$

is the tangent curve at point of inflection.

Lag time:

After some time, at a special point the growth may reach its maximum and pauses. This point is called as lag time and denoted by λ. This point is the intercept of apsis of the tangent curve at inflection point.

That is, by writing the tangent equation at $(t_{\text{inf}}, y(t_{\text{inf}}))$ for the intercept of apsis coordinate,

$$-y(t_{\text{inf}}) = y'(t_{\text{inf}}) \cdot (\lambda - t_{\text{inf}})$$

can be reached. Performing some manipulations on this equation, we can find

$$-y_{\infty}\left(k + \ln t_0 + 1\right) + y\left(0\right)\left(k + \ln t_0\right) = \frac{\left(k + \ln t_0\right)\ \left(y_{\infty} - y\left(0\right)\right)}{e^{-(k+1)}}\ (\lambda\ - e^{-(k+1)}),$$

$$\frac{-y_\infty (k+\ln t_0+1)+y(0)(k+\ln t_0)}{(k+\ln t_0)\ (y_\infty - y(0))} = \frac{\lambda}{e^{-(k+1)}} - 1,$$

$$\frac{-y_\infty (k+\ln t_0+1)+y(0)(k+\ln t_0)}{(k+\ln t_0)\ (y_\infty - y(0))} + 1 = \frac{\lambda}{e^{-(k+1)}},$$

$$\frac{-y_\infty (k+\ln t_0+1)+y(0)(k+\ln t_0)+(k+\ln t_0)\ (y_\infty - y(0))}{(k+\ln t_0)\ (y_\infty - y(0))} = \frac{\lambda}{e^{-(k+1)}}.$$

Then, reducing the formula

$$\frac{y_\infty}{(k+\ln t_0)\ (y(0) - y_\infty)} = \frac{\lambda}{e^{-(k+1)}}.$$

Can be found. Then the lag time parameter λ is given as follows:

$$\lambda = \frac{y_\infty\ e^{-(k+1)}}{(k+\ln t_0)\ (y(0) - y_\infty)}.$$

4. Conclusion

Growth functions are mathematical models that allow the calculation / prediction of the increase in the population, the growth or life span of a living thing, length-weight relationships of individuals etc. In order to best forecast growth, increase or decrease, scientists continue to work on these models and try to add new and comprehensive models to the literature. This study has emerged with a similar purpose. In this study, a new mathematical growth model, L-MMH, which is in the structure of the basic models given by Michaelis and Menten (1913) and Hill (1910, 1913) but includes a logarithmic function, is introduced and some of its determining properties are given. In future studies, it will be possible to predict growth using L-MMH model on real data sets.

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