J COMP SCI
APPL MATH
Journal of Computer Science and Applied Mathematics
Vol. 3, no.1, (2021), 23–36
ISSN: 1857-9582 https://doi.org/10.37418/jcsam.3.1.4

THE DISTANCE RELATED SPECTRA OF SOME SUBDIVISION RELATED GRAPHS

Indulal Gopalapillai[1] and Deena C. Scaria

ABSTRACT. Let G be a connected graph with a distance matrix D. The distance eigenvalues of G are the eigenvalues of D, and the distance energy $E_D(G)$ is the sum of its absolute values. The transmission $Tr(v)$ of a vertex v is the sum of the distances from v to all other vertices in G. The transmission matrix $Tr(G)$ of G is a diagonal matrix with diagonal entries equal to the transmissions of vertices. The matrices $D^L(G) = Tr(G) - D(G)$ and $D^Q(G) = Tr(G) + D(G)$ are, respectively, the Distance Laplacian and the Distance Signless Laplacian matrices of G. The eigenvalues of $D^L(G)$ ($D^Q(G)$) constitute the Distance Laplacian spectrum (Distance Signless Laplacian spectrum). The subdivision graph $S(G)$ of G is obtained by inserting a new vertex into every edge of G. We describe here the Distance Spectrum, Distance Laplacian spectrum and Distance Signless Laplacian spectrum of some types of subdivision related graphs of a regular graph in the terms of its adjacency spectrum. We also derive analytic expressions for the distance energy of $\bar{S}(C_p)$, partial complement of the subdivision of a cycle C_p and that of $\overline{S(C_p)}$, complement of the even cycle C_{2p}.

[1]*corresponding author*

Key words and phrases. Distance Matrix, Distance Laplacian matrix, Distance Signless Laplacian matrix, Distance spectrum, Distance Energy.

1. Introduction

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. The distance matrix or $D-$ matrix, D of G is defined as $D[G] = [d_{ij}]$, where d_{ij} is the distance between the vertices v_i and v_j in G. We denote by $Tr(G)$, a diagonal matrix whose i^{th} diagonal entry is $d(v_i)$ which is the transmission of the vertex $v_i \in V$, defined as the sum of the distance from v_i to the remaining vertices of G. The Distance Laplacian matrix and the Distance Signless Laplacian matrix of the graph G, are defined as $D^L(G) = Tr(G) - D(G)$ and $D^Q(G) = Tr(G) + D(G)$ respectively. The matrices $D(G)$, $D^L(G)$ and $D^Q(G)$ are all symmetric, nonnegative and irreducible.Hence all the eigenvalues are real and can be ordered. The eigenvalues and spectrum of $D(G)$(respectively, $D^L(G)$, $D^Q(G)$) are said to be the distance eigenvalues(respectively, Distance Laplacian eigenvalues and Distance Signless Laplacian eigenvalues), in short $D-$ eigenvalues and distance spectrum (respectively, Distance Laplacian spectrum, Distance Signless Laplacian spectrum), in short $D-$ spectrum, of G, respectively. The distance energy $E_D(G)$ of G is defined as the sum of the absolute values of its distance eigenvalues.

Researchers have been studying distance eigenvalues of graphs for many years. In the literature, there are very few graphs whose full distance spectrum is known. Ruzieh and Powers [4] found all the eigenvalues and eigenvectors of the distance matrix of path P_n on n vertices. In [5], Graovac et al. determined all the distance eigenvalues of the cycle C_n with n vertices. In [6], Atik and Panigrahi found the distance spectrum of some distance regular graphs including the well known Johnson graphs. In [8], the authors describe the distance spectrum of the subdivision vertex join and subdivision edge join of two regular graphs and in [9], the Distance Laplacian and Distance Signless Laplacian spectrum of the subdivision vertex join and subdivision edge join of two regular graphs are studied. For more results related to $D-$ spectrum of graphs, readers may see the survey [7] and [10, 11]. Also for the numerous applications of distance matrix in chemistry and other branches of sciences please see [1] and the papers cited therein.

In [3] the adjacency spectrum of the partial complement of the subdivision graph of a regular graph G in terms of that of G is derived. A similar work on the distance spectrum is not explored, maybe due to the fact that the distance matrix of the subdivision graph, in most of the cases cannot be expressed in a workable block form. In the case of complements of regular graphs G, the adjacency spectrum can be obtained from the corresponding information of that of G, whereas the distance spectrum of graph complements is not studied in detail in the literature.

In this paper we obtain the three kind of spectra of the partial complement of the subdivision graph of a regular graph G, complement of the subdivision graph of a regular graph and the subdivision graph of K_p. We also derive the distance energy of $\bar{S}(C_p)$, partial complement of the subdivision of a cycle C_p and the distance energy of $\overline{S\,(C_p)}$, complement of the subdivision of a cycle C_p.

The discussions in this paper are based upon the following lemmas and definition:

Lemma 1.1. *Let G be an $r-$ regular (p,q) graph with an adjacency matrix A and an incidence matrix R. Let $L(G)$ be its line graph. Then $RR^T = A + rI$, $R^TR = A(L(G)) + 2I$.*

Lemma 1.2. [2] *Let G be an $r-$ regular (p,q) graph with $spec(G) = \{r, \lambda_2, \ldots, \lambda_p\}$. Then*

$$spec(L(G)) = \begin{pmatrix} 2r-2 & \lambda_2 + r - 2 & \cdots & \lambda_p + r - 2 & -2 \\ 1 & 1 & \cdots & 1 & q-p \end{pmatrix}.$$

Lemma 1.3. *Let G be an $r-$ regular (p,q) graph with an adjacency matrix A, an incidence matrix R and an adjacency matrix B of its line graph. Let X be an eigenvector of G corresponding to the eigenvalue $-r$ if it exists and Y be an eigenvector of $L(G)$ corresponding to the eigenvalue -2. Then*

1. *$R^TX = 0$ and RY=0*

2. *If Z is an eigenvector of A corresponding to the eigenvalue λ, then it is an eigenvector of RR^T corresponding to the eigenvalue $\lambda + r$.*

Definition 1.1. *[3] Let G be a (p,q) graph. Corresponding to every edge e of G introduce a vertex and make it adjacent with all the vertices not incident with e in G. Delete the edges of G only. The resulting graph is called the **partial complement of the subdivision graph (PCSD)** of G denoted by $\overline{S}(G)$.*

Figure 1 illustrates the PCSD of C_5.

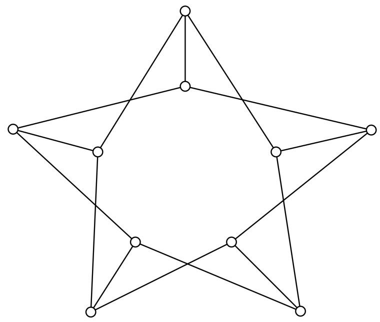

FIGURE 1. $\overline{S}(C_5)$

The spectrum of an $n \times n$ symmetric matrix with k distinct eigenvalues λ_i , $i = 1, 2, \ldots, k$ with multiplicities m_i , $i = 1, 2, \ldots, k$ is represented as

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}$$

2. THE DISTANCE SPECTRUM OF THE PARTIAL COMPLEMENT OF THE SUBDIVISION GRAPH $(PCSD)$ OF A REGULAR GRAPH G

In this section we obtain the Distance Spectrum of the $PCSD$ of a regular graph G of order $p \geq 5$.

Theorem 2.1. *Let G a connected $r-$ regular (p, q) graph with adjacency spectrum $\{r, \lambda_2, \ldots, \lambda_p\}$. Then*

$$spec_D(\bar{S}(G))$$
$$= \begin{pmatrix} (p+q-2) \pm \sqrt{p^2+q^2-pq+2(pr+2q+4r)} & 2(-1 \pm \sqrt{\lambda_i + r}\ ) & -2 \\ 1 & 1 & q-p \end{pmatrix},$$

i=2,3,...,p.

Proof. Let G be an $r-$ regular graph on p vertices and q edges with $p \geq 5$ and an adjacency matrix A(G) with eigenvalues $\{r, \lambda_2, \ldots \lambda_p\}$. Let R be the $p \times q$, vertex - edge incidence matrix. Let $v_1, v_2, \ldots v_p$ be the vertices and $e_1, e_2, \ldots e_q$ be those vertices corresponding to the q edges of G. Let H denote the $PCSD$ of G. Then from Definition 1.1, it follows easily that

$$d(v_i, v_j) = 2$$

$$d(v_i, e_k) = \begin{cases} 3, & \text{if } v_i \text{ is incident with } e_k \\ 1, & \text{otherwise} \end{cases}$$

$$d(e_k, e_l) = 2 \text{ holds in } \; H.$$

Thus the distance matrix D of H is of the form $D = \begin{bmatrix} 2(J-I)_p & J_{p\times q} + 2R \\ J_{q\times p} + 2R^T & 2(J-I)_q \end{bmatrix}$.
Now we find the distance spectrum of H as follows.

Let X_i, $i = 1, 2, \ldots, q-p$ be the set of $q-p$ linearly independent eigenvectors corresponding to the eigenvalue -2 of $L(G)$. Since $L(G)$ is regular $J_{1\times q}$ is orthogonal to X_i and $R\,.\,X_i = 0$. Now we see that $\phi_i = \begin{bmatrix} 0_{p\times 1} \\ X_i \end{bmatrix}$, $i = 1, 2, \ldots q-p$ are eigenvectors of D corresponding to -2. This is because

$$\begin{aligned} D.\, \phi_i &= \begin{bmatrix} 2(J-I)_p & J_{p\times q} + 2R \\ J_{q\times p} + 2R^T & 2(J-I)_q \end{bmatrix} \begin{bmatrix} 0_{p\times 1} \\ X_i \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -2X_i \end{bmatrix} = -2\,.\,\phi_i \end{aligned}$$

Now let Y_i be an eigenvector of A corresponding to the eigenvalue $\lambda_i \neq r$. Then there are two cases.

Case 1. $\lambda_i \neq -r$

In this case $R^T \cdot\; Y_i \neq 0$, otherwise $RR^T \cdot\; Y_i = 0$ yield Y_i is an eigenvector corresponding to $-r$ of A.

Then: $\varphi_i = \begin{bmatrix} \sqrt{\lambda_i + r} \cdot\; Y_i \\ R^T \cdot\; Y_i \end{bmatrix}$ is an eigenvector of D corresponding to the eigenvalue $2(-1 + \sqrt{\lambda_i + r}\;)$. This is because

$$\begin{aligned} D.\, \varphi_i &= \begin{bmatrix} 2(J-I)_p & J_{p\times q} + 2R \\ J_{q\times p} + 2R^T & 2(J-I)_q \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_i + r} \cdot Y_i \\ R^T \cdot Y_i \end{bmatrix} \\ &= \begin{bmatrix} -2\sqrt{\lambda_i + r} \cdot Y_i + 2(\lambda_i + r) \cdot Y_i \\ 2(-1 + \sqrt{\lambda_i + r})\; R^T \cdot Y_i \end{bmatrix} = 2(-1 + \sqrt{\lambda_i + r}\;) \cdot \varphi_i. \end{aligned}$$

Similarly we can prove that $\varphi_i = \begin{bmatrix} -\sqrt{\lambda_i + r} \cdot Y_i \\ R^T \cdot Y_i \end{bmatrix}$ is an eigenvector of D corresponding to the eigenvalue $2(-1 - \sqrt{\lambda_i + r}\ )$.

Case 2. $\lambda_i = -r$. Then by lemma 1.2, $R^T Y = 0$.

Then: $\varphi_i = \begin{bmatrix} Y_i \\ 0 \end{bmatrix}$ is an eigenvector of D corresponding to the eigenvalue -2. For

$$\begin{aligned} D \cdot \varphi_i &= \begin{bmatrix} 2(J-I)_p & J_{p\times q} + 2R \\ J_{q\times p} + 2R^T & 2(J-I)_q \end{bmatrix} \begin{bmatrix} Y_i \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -2 \cdot Y_i \\ 0 \end{bmatrix} = -2 \cdot \varphi_i. \end{aligned}$$

Thus, we have altogether constructed $p+q-2$ eigenvectors and corresponding eigenvalues and there remains two. By the very construction, all are linearly independent. The already constructed eigenvectors are orthogonal to $\begin{bmatrix} J_{p\times 1} \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ J_{q\times 1} \end{bmatrix}$. Therefore, these two vectors span the remaining two eigenvectors of D. Thus, they are of the form $\varsigma = \begin{bmatrix} \alpha J_{p\times 1} \\ \beta J_{q\times 1} \end{bmatrix}$ for some $(\alpha, \beta) \neq (0,0)$. Therefore, if ω is an eigenvalue of D with an eigenvector $\begin{bmatrix} \alpha J_{p\times 1} \\ \beta J_{q\times 1} \end{bmatrix}$, from $D \begin{bmatrix} \alpha J_{p\times 1} \\ \beta J_{q\times 1} \end{bmatrix} = \omega \begin{bmatrix} \alpha J_{p\times 1} \\ \beta J_{q\times 1} \end{bmatrix}$, we get the system of equations:

$$2(p-1)\alpha + (q+2r)\beta = 0$$

$$(p+4)\alpha + 2(q-1)\beta = 0$$

Then ς is a non- trivial solution to the above system of equations, which in turn are the eigenvalues of the matrix $\begin{bmatrix} 2(p-1) & q+2r \\ p+4 & 2(q-1) \end{bmatrix}$, whose characteristic equation is $x^2 - 2(p+q-2)x + 3pq - 2pr - 4p - 8q - 8r + 4 = 0$ and this completes the proof. $\square$

Note: The case when $p = 1, 2\ and\ 3$ are not admissible as the $PCSD$ in these cases are not connected. When $p = 4$ and $r = 2$, G is C_4 and the $PCSD$ is

C_8 whose distance spectrum is already known. When $p = 4$ and $r = 3$,G is K_4 and the $PCSD$ is the subdivision graph of K_4, the distance spectrum of which is given in Theorem 5.4.

3. The Distance Laplacian Spectrum of the partial complement of the subdivision graph $(PCSD)$ of a regular graph G

In this section, the Distance Laplacian Spectrum of the $PCSD$ of a regular graph G of order $p \geq 5$ is obtained.

Theorem 3.1. *Let G be a connected r- regular (p, q) graph with an adjacency matrix $A(G)$ whose spectrum is $(r, \lambda_2, \ldots \lambda_p)$. Then the Distance Laplacian spectrum of the $PCSD$ of G is*

$$\begin{pmatrix} p+2q+2r & \dfrac{(p+q+4r)\pm\alpha}{2} & \dfrac{(3p+3q+4r)\pm\beta}{2} \\ q-p & 1 & 1 \end{pmatrix}, i = 2, 3, \ldots, p$$

where $\alpha = \sqrt{(p+q)^2 + 8(pr + 2q + 4r)}$ *and* $\beta = \sqrt{(p-q)^2 + 16\left(\lambda_i + r\right)}$

Proof. The Distance Laplacian matrix D^L of H is of the form

$$D^L = \begin{bmatrix} (2p+q+2r)I - 2J & -J - 2R \\ -J - 2R^T & (p+2q+2r)I - 2J \end{bmatrix}.$$

Now we find the Distance Laplacian spectrum of H as follows.

Let X_i, $i = 1, 2, \ldots q-p$ be the set of $q-p$ linearly independent eigenvectors corresponding to the eigenvalue -2 of $L(G)$. Since $L(G)$ is regular, $J_{1\times q}$ is orthogonal to X_i and by Lemma 1.3 $R\dot{X}_i = 0$. Now we see that $\Theta_i = \begin{bmatrix} 0_{p\times 1} \\ X_i \end{bmatrix}$, $i = 1, 2, \ldots q-p$, are eigenvectors of D^L.

For

$$D^L.\, \Theta_i = \begin{bmatrix} (2p+q+2r)I - 2J & -J - 2R \\ -J - 2R^T & (p+2q+2r)I - 2J) \end{bmatrix} \begin{bmatrix} 0 \\ X_i \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ (p+2q+2r)X_i \end{bmatrix} = (p+2q+2r)\ \cdot \Theta_i.$$

Let $\lambda_i \neq r$ be an eigenvalue of A with an eigenvector X_i. We investigate the condition under which $\chi_i = \begin{bmatrix} tX_i \\ R^T X_i \end{bmatrix}$ is an eigenvector corresponding μ of D^L. For from $D^L \cdot \chi_i = \mu \cdot \chi_i$, we get

$$\begin{bmatrix} (2p+q+2r)I - 2J & -J-2R \\ -J-2R^T & (p+2q+2r)I-2J \end{bmatrix} \begin{bmatrix} tX_i \\ R^T X_i \end{bmatrix} = \mu \begin{bmatrix} tX_i \\ R^T X_i \end{bmatrix}$$

$$(2p+q+2r)t - 2(\lambda_i + r) = \mu t \tag{1}$$

$$-2t + (p+2q+2r) = \mu. \tag{2}$$

Eliminating μ from (1) and (2), we get

$$2t^2 + (p-q)t - 2(\lambda_i + r) = 0.$$

So that t has two values

$$t_1 = \frac{-(p-q) + \sqrt{(p-q)^2 + 16(\lambda_i + r)}}{4},$$
$$t_2 = \frac{-(p-q) - \sqrt{(p-q)^2 + 16(\lambda_i + r)}}{4}.$$

Thus we get $-2t_1 + (p+2q+2r)$ and $-2t_2 + (p+2q+2r)$ for $i = 2, 3, \ldots, p$ as eigenvalues and there remains two. The two remaining eigenvectors of D^L are then of the form $\tau = \begin{bmatrix} \alpha J \\ \beta J \end{bmatrix}$ for some $(\alpha, \beta) \neq (0,0)$. Now suppose that ϱ is an eigenvalue of D^L with an eigenvector τ. Then from $D^L\, \tau = \varrho\, \tau$, we can see that the remaining two are the eigenvalues of the matrix $\begin{bmatrix} q+2r & -q-2r \\ -p-4 & p+2r \end{bmatrix}$, whose characteristic equation is $x^2 - (p+q+4r)x + 2(2r^2 + qr - 2q - 4r) = 0$ and this completes the proof. $\square$

4. The Distance Signless Laplacian Spectrum of the partial complement of the subdivision graph $(PCSD)$ of a regular graph G

In this section, the Distance Signless Laplacian Spectrum of the $PCSD$ of a regular graph G of order $p \geq 5$ is obtained.

The proof of the following theorem is on similar lines as that of Theorem 3.1

Theorem 4.1. *let G a connected r- regular (p, q) graph with an adjacency matrix $A(G)$ whose spectrum is $(r, \lambda_2, \dots \lambda_p)$. Then the Distance Signless Laplacian spectrum of the $PCSD$ of a regular graph G is*

$$\begin{pmatrix} p+2q+2r-4 & \frac{(5p+5q+4r-8)\pm\alpha}{2} & \frac{(3p+3q+4r-8)\pm\beta}{2} \\ q-p & 1 & 1 \end{pmatrix}$$

$i = 2, 3, \dots, p$ *where*

$$\alpha = \sqrt{(3p-3q)^2 + 4(pq+2pr+4q+8r)} \quad \textit{and} \quad \beta = \sqrt{(p-q)^2 + 16(\lambda_i + r)}.$$

5. Distance spectrum of the complement of the subdivision graph of a regular graph

In this section, the Distance Spectrum of the complement of the subdivision graph of a regular graph is obtained.

Theorem 5.1. *Let G be an $r-$regular (p, q) graph with adjacency spectrum $\{r, \lambda_2, \dots, \lambda_p\}$. Then Distance spectrum of the complement of the subdivision graph of G is*

$$\begin{pmatrix} \frac{(p+q-2)\pm\sqrt{(p+q)^2+4(pr+2q+2r)}}{2} & (-1\pm\sqrt{\lambda_i+r}\,) & -1 \\ 1 & 1 & q-p \end{pmatrix},$$

$i = 2, 3, \dots, p.$

Proof. Let G be an $r-$ regular graph with p vertices and an adjacency matrix $A(G)$. Let R be the incidence matrix of G. Let F denote the complement of the subdivision graph of G. Then by a proper labelling of vertices in F, it follows easily that

$$d(v_i, v_j) = 1$$

$$d(v_i, e_k) = \begin{cases} 2, & \text{if } v_i \text{ is incident with } e_k \\ 1, & \text{otherwise} \end{cases}$$

$$d(e_k, e_l) = 1.$$

Thus the distance matrix D of F has the form $D = \begin{bmatrix} (J-I)_p & J_{p\times q}+R \\ J_{q\times p}+R^T & (J-I)_q \end{bmatrix}$.

Now the rest of the proof is along similar lines as that of previous theorems. □

Now we present the following theorems, the proofs of which can be similarly obtained.

Theorem 5.2. *Let G a connected r- regular (p, q) graph with adjacency spectrum $(r, \lambda_2, \dots \lambda_p)$. Then the Distance Laplacian spectrum of the complement of the subdivision graph of a regular G is*

$$\begin{pmatrix} p+q+r & \dfrac{(p+q+2r) \pm \sqrt{(p+q)^2+4(pr+2q+2r)}}{2} & p+q+r \pm \sqrt{\lambda_i + r} \\ q-p & 1 & 1 \end{pmatrix},$$

$i = 2, 3, \dots, p$.

Theorem 5.3. *Let G a connected r- regular (p, q) graph with adjacency spectrum $(r, \lambda_2, \dots \lambda_p)$. Then the Distance Signless Laplacian spectrum of the complement of the subdivision graph of G consists of the numbers:*

$$\begin{pmatrix} p+q+r-2 & \Delta & p+q+r-2 \pm \sqrt{\lambda_i + r} \\ q-p & 1 & 1 \end{pmatrix},$$

$\Delta = \dfrac{(3p+3q+2r-4) \pm \sqrt{(p+q)^2+4(pr+2q+2r)}}{2}$, $i = 2, 3, \dots, p$.

Theorem 5.4. *Let A be the adjacency matrix of K_p with spectrum $\begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix}$. Then the Distance spectrum of the subdivision graph of K_p is*

$$\begin{pmatrix} (p+2q-2r-1) \pm \alpha & 0 & -2(p-1) \\ 1 & \dfrac{p^2-p-2}{2} & p-1 \end{pmatrix}$$

where $\alpha = \sqrt{(2q-2r)^2+p^2+5pq-2pr-2p-8q+4r+1}$ *and* $r = p-1$ *and* $q = \dfrac{p(p-1)}{2}$.

Theorem 5.5. *Let A be the adjacency matrix of K_p with spectrum* $\begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix}$. *Then the Distance Laplacian spectrum of the subdivision graph of K_p is*

$$\begin{pmatrix} 4q-3p+6 & \dfrac{(3p+3q-4r)\pm\alpha}{2} & \dfrac{(7q-p+4)\pm\beta}{2} \\ q-p & 1 & p-1 \end{pmatrix},$$

where $\sqrt{(3p+3q)^2-8(3pr-4r+6q)}$, $\beta = \sqrt{(q-p)^2+16(p-2)}$ *and* $r = p-1$ *and* $q = \dfrac{p(p-1)}{2}$.

Theorem 5.6. *Let A be the adjacency matrix of K_p with spectrum* $\begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix}$. *Then the Distance Signless Laplacian spectrum of the subdivision graph of K_p is*

$$\begin{pmatrix} 4q-3p+6 & \dfrac{7p+11q-12r-4\pm\alpha}{2} & \Delta \\ q-p & 1 & p-1 \end{pmatrix},$$

where $\Delta = \dfrac{(7q-5p+8)\pm\sqrt{(q-5p+12)^2+16(p-2)}}{2}$,

$$\alpha = \sqrt{(p+5q)^2-8(p+q+4r-2pq+10qr+pr-8r^2-2)},$$

$r = p-1$ *and* $q = \dfrac{p(p-1)}{2}$.

6. The Distance Energy of two graphs related to the cycle C_p

Theorem 6.1. *For $p \geq 6$,*

$$E_D\left(\overline{S}(C_p)\right) = \begin{cases} 4\sqrt{3}\ \cot\dfrac{\pi}{2p} + \dfrac{4}{3}(4p-9); p \equiv 0\ (b \mod 3) \\ 8\cos\left(\dfrac{p-1}{3}\times\dfrac{\pi}{2p}\right)\csc\dfrac{\pi}{2p} + \dfrac{4}{3}(4p-10)\,; p\equiv 1\ (\mod 3) \\ 8\cos\left(\dfrac{p+1}{3}\times\dfrac{\pi}{2p}\right)\csc\dfrac{\pi}{2p} + \dfrac{4}{3}(4p-8)\,; p\equiv 2\ (\mod 3) \end{cases}.$$

Proof. By Theorem 2.1 we have

$$Spec_D\ (\bar{S}(C_p)) = \begin{pmatrix} p-6 & 3p+2 & 2(-1\pm\ 2\cos\dfrac{\pi j}{p}) \\ 1 & 1 & 1 \end{pmatrix}, j = 1, 2\ldots, p-1.$$

The distance energy of $\bar{S}(C_p)$ is the sum of the absolute values of all of its eigenvalues which in turn is twice of the sum of its positive eigenvalues. Therefor first we find the positive eigenvalues other than $p-6$ and $3p+2$.

For this we consider the following three cases.

Case 1. $p \equiv 0 \mod 3)$

The numbers $2(-1+\ 2\cos\frac{\pi j}{p})$ are positive only for $\frac{\pi}{p}j \leq \ \frac{\pi}{3}$ and consequently for $j=1,2,\ldots,\frac{p}{3}$. Thus the positive values from $2(-1+\ 2\cos\frac{\pi j}{p})$ are $2(-1+\ 2\cos\frac{\pi}{p})$, $2(-1+\ 2\cos\frac{2\pi}{p}),\ldots,2(-1+\ 2\cos(\frac{\pi}{p}\times\frac{p}{3}))$.

Let $C=2\cos\frac{\pi}{p}+2\cos\frac{2\pi}{p}+\cdots+2\cos(\frac{\pi}{p}\times\frac{p}{3})$ and $S=2\sin\frac{\pi}{p}+2\sin\frac{2\pi}{p}+\cdots+2\sin(\frac{\pi}{p}\times\frac{p}{3})$.

So $C+i\ S=2\{\alpha+\alpha^2+\cdots+\alpha^{\frac{p}{3}}\}=2\alpha\dfrac{1-\alpha^{\frac{p}{3}}}{1-\alpha}$, where $\alpha=e^{i\pi/p}$. Equating real parts, we get $C=\dfrac{-1}{2}+\dfrac{\sqrt{3}}{2}\cot\dfrac{\pi}{2p}$. The positive contribution from $(-1+\ 2\cos\frac{\pi j}{p})$ is $\dfrac{-p}{3}+\dfrac{-1}{2}+\dfrac{\sqrt{3}}{2}\cot\dfrac{\pi}{2p}$. The numbers $2(-1-\ 2\cos\frac{\pi j}{p})$ are positive only for $\frac{\pi}{p}j\geq\ \frac{2\pi}{3}$. Then the positive values of $2(-1-\ 2\cos\frac{\pi j}{p})$ are $2(-1-\ 2\cos\frac{2\pi}{3})$, $2(-1-\ 2\cos(\frac{2\pi}{3}+\frac{\pi}{p})),\ldots,2(-1-\ 2\cos(\frac{2\pi}{3}+(\frac{p}{3}-1)\times\frac{\pi}{p}))$.

Let $C=-2\cos\frac{2\pi}{3}-\ 2\cos(\frac{2\pi}{3}+\frac{\pi}{p})-\cdots-\ 2\cos(\frac{2\pi}{3}+(\frac{p}{3}-1)\times\frac{\pi}{p})$ and $S=-2\sin\frac{2\pi}{3}-\ 2\sin(\frac{2\pi}{3}+\frac{\pi}{p})-\cdots-\ 2\sin(\frac{2\pi}{3}+(\frac{p}{3}-1)\times\frac{\pi}{p})$. So $C+iS=-2\ e^{i2\pi/3}\{1+\alpha+\alpha^2+\cdots+\alpha^{\frac{p}{3}-1}\}=-2\ e^{i2\pi/3}\ \dfrac{1-\alpha^{\frac{p}{3}}}{1-\alpha}$.

Equating real parts, we get $C=\dfrac{-1}{2}+\dfrac{\sqrt{3}}{2}\cot\dfrac{\pi}{2p}$. Therefor the positive contribution from $(-1-\ 2\cos\frac{\pi j}{p})$ is $\dfrac{-p}{3}+\dfrac{-1}{2}+\dfrac{\sqrt{3}}{2}\cot\dfrac{\pi}{2p}$. Therefore the total positive contributions from these eigenvalues $=2\{\dfrac{-2p}{3}-1+\sqrt{3}\cot\dfrac{\pi}{2p}\}$. Distance Energy

$= 2\times$ sum of positive eigenvalues $= 2 \times \{p - 6 + 3p + 2 + 2\sqrt{3}\cot\frac{\pi}{2p} - \frac{4p}{3} - 2\}$.
$E_D(\bar{S}(C_p))$=$4\sqrt{3}\cot\frac{\pi}{2p} - \frac{4}{3}(4p-9)$.

The other two cases $p \equiv 1 \mod 3)$ and $p \equiv 2 \mod 3)$ can be similarly proved.

□

The proof of the following theorem is along similar lines as that of Theorem 6.1.

Theorem 6.2.

$$E_D\left(\overline{S\left(C_p\right)}\right) = \begin{cases} 2\sqrt{3}\ \cot\frac{\pi}{2p} + \frac{8p}{3}\ ; p \equiv 0 \mod 3) \\ 4\cos(\frac{p-1}{3} \times \frac{\pi}{2p}) cosec\frac{\pi}{2p} + \frac{2}{3}(4p-1); p \equiv 1 \mod 3) \\ 4\cos(\frac{p+1}{3} \times \frac{\pi}{2p}) cosec\frac{\pi}{2p} + \frac{2}{3}(4p+1); p \equiv 2 \mod 3) \end{cases}.$$

7. Conclusion

In this paper, we considered distance related spectrum of some subdivision related graphs. We propose the computation of distance related spectrum of some other classes of graphs related to subdivision graphs.

References

[1] K.Balasubramanian: *Computer generation of distance polynomials of graphs*, Journel of Computational Chemistry, **11** (1990), 829–836.

[2] D. Cvetkovic, M. Doob, H. Sachs: *Spectra of Graphs– Theory and Application*, Johann Ambrosius Barth Verlag, 3^{rd} edition, 1995.

[3] G.Indulal, A. Vijayakumar: *A note on energy of some graphs*, MATCH Commun. Math. Comput. Chem., **59**(2) (2008), 269-271.

[4] S.N. Ruzieh, D.L. Powers: *The distance Spectrum of the path P_n and the first distance eigenvector of connected graphs*, Linear Multilinera Algebra, **28** (1990), 75-81.

[5] A. Graovac, G.Jashari, M. Strunje: *On the distance spectrum of a cycle*, Aplikace matematiky, **30**(4) (1985), 286-290.

[6] F. Atik, P. Panigrahi: *On the distance spectrum of distance regular graphs*, Linear Algebra Appl., bf478 (2015). 256-273.

[7] M. Aouchiche, P. Hansen: *Distance spectra of graphs : A survey*, Linear Algebra Appl., **458** (2014), 301-384.

[8] G. Indulal, Deena C. Scaria, X. Liu: *The Distance spectrum of the subdivision vertex join and subdivision edge join of two regular graphs*, Discrete Math. Lett. **1** (2019), 36-41.

[9] D.C. Scaria, G. Indulal: *The Distance Laplacian and Distance Signless Laplacian Spectrum of the subdivision vertex join and subdivision edge join of two regular graphs*, Discrete Mathematics Algorithms and Applications, **11**(5) (2019), art.no. 1950053.

[10] S. Sun, Z. Wan: *On the Change of Distance Energy of Complete Bipartite Graph due to Edge Deletion*, Journal of Mathematics **2021** (2021), 1-3.

[11] Z.Tang, R. Wu, H. Chen, H. Deng: *The distance spectrum of two new operations of graphs*, Transactions on Combinatorics, **9**(3) (2020), 125-138.

Department of Mathematics
St.Aloysius College, Edathua
Alappuzha - 689573, India.
Email address: `indulalgopal@gmail.com`

Department of Mathematics
St.Aloysius College, Edathua
Alappuzha - 689573, India and
Marthoma College, Thiruvalla
Pathanamthitta - 689103, India.
Email address: `deenacelinescaria@gmail.com`