J COMP SCI
APPL MATH
Journal of Computer Science and Applied Mathematics
Vol. 3, no.2, (2021), 37–57
ISSN: 1857-9582 https://doi.org/10.37418/jcsam.3.2.1

ON THE WIENER INDEX OF F_H SUMS OF GRAPHS

Liju Alex and Indulal Gopalapillai[1]

ABSTRACT. Wiener index is the first among the long list of topological indices which was used to correlate structural and chemical properties of molecular graphs. In [5] M. Eliasi, B. Taeri defined four new sums of graphs based on the subdivision of edges with regard to the cartesian product and computed their Wiener index. In this paper, we define a new class of sums called F_H sums and compute the Wiener index of the resulting graph in terms of the Wiener indices of the component graphs so that the results in [5] becomes a particular case of the Wiener index of F_H sums for $H = K_1$, the complete graph on a single vertex.

1. INTRODUCTION

A simple graph G is connected if every pair of vertices are connected by a path. The distance $d(u, v)$ between any two vertices in a connected graph is the length (number of edges) of the shortest path between them. The concept of distance in graph is of vital importance as it is the basic tool to study the topological aspects of graphs, one among them is the Wiener index named after H. Wiener [11]. Wiener index of a graph G denoted by $W(G)$ is defined as the sum of the distance between all pairs of vertices on a connected graph.

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d(u, v).$$

[1] *corresponding author*

Key words and phrases. Wiener index ($W(G)$), F_H sums.

The origin of Wiener index itself comes from a study on relationship between boiling points and structural aspects of paraffin molecules [11]. Later in molecular graph theory, Wiener index found its supreme importance in studying structural as well as physical composition of various chemical graphs [1]. In [3] H Hosoya introduced a polynomial associated with the Wiener index called Wiener polynomial and obtained the Wiener index as the derivative of this polynomial at unity. Later this polynomial was renamed as Hosoya polynomial. For furthur results about Wiener index and Hosoya polynomial see [1, 10].

Computing the topological indices of various graph operations has been a subject of recent research. Y. N Leh, I. Gutman computed the Wiener index of various graph operations such as product, join, composition [13]. D Stevanović computed the Wiener polynomial of product, join, composition by generalising the earlier results [10]. Cvetković proposed four new graphs S, R, Q, T based on subdivision of edges in [2]. In [12] W Yan *et al* computed the Wiener index of each subdivision graphs S, R, Q, T in terms of the parent graph as well as the other subdivision related graphs. In [5] M. Eliasi, B. Taeri defined a new operation on graphs called F sums based on these four subdivision graphs and computed the Wiener index of F sum in terms of the Wiener index of the component graphs. H Deng *et al* computed the Zagreb indices of F sums [6] and S. Akhtar, M Imran computed the forgotten index of F sums [9]. The generalised version of F sums called generalized F_k sums was introduced by J.B Liu *et al* and they computed the Zagreb indices of the sums in terms of its factor graphs [7]. Based on the subdivisions $S.R, Q, T$ of a graph G, four subdivisions with respect to a graph H (S_H, R_H, Q_H, T_H) can be proposed by introducing a new graph H corresponding to each edge of the parent graph and joining the endvertices of each edge to all vertices to the corresponding copy of H. The graph $S_H(G)$ is the edge corona of G and H and all other subdivisions are analogous versions of R, Q, T. In this paper we obtain the inter relationship of Wiener index of four graphs S_H, R_H, Q_H and T_H. We also define a new sum called F_H sums, an analogous version of F sum in terms of $F_H \in \{S_H, R_H, Q_H, T_H\}$ and compute the Wiener index of F_H sums. Thus, the results established in [5] will be particular case for $(H = K_1)$ of the results in this paper.

2. F_H - SUMS

Let G_1, G_2 be two graphs with vertex set $V_1(G)$, $V_2(G)$ and edge set $E_1(G)$ and $E_2(G)$ respectively. Let H be any graph. Then the four graphs associated with H are $S_H(G_1), R_H(G_1), Q_H(G_1), T_H(G_1)$ and are as defined as follows:

(1) $S_H(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 with a copy of H and making every vertex in the ith copy of H adjacent to the end vertices of e_i for each $e_i \in E(G_1)$. That is, $S_H(G_1)$ is a graph with vertex set $V(S_H(G_1)) = V(G_1)\bigcup V_e(H)$ where $V_e(H) = \cup_{i=1}^{|E(G_1)|}V_i(H)$, $V_i(H) = V(H)\ \forall i$ and the edge set $E(S_H(G_1)) = \{(v,h), (u,h) : e = vu \in E(G_1), h \in V_e(H)\} \cup E_e(H)$ where

$$E_e(H) = \cup_{i=1}^{|E(G_1)|}E_i(H), \quad E_i(H) = E(H), \quad \forall i.$$

(2) $R_H(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 with a copy of H and making every vertex in the ith copy of H adjacent to the end vertices of e_i for each $e_i \in E(G_1)$ also keeping every edge in G_1 as well. That is, $R_H(G_1)$ is a graph with vertex set $V(R_H(G_1)) = V(G_1)\bigcup V_e(H)$ and edge set $E(R_H(G_1)) = \{(v,h),(u,h) : e = vu \in E(G_1), h \in V_e(H)\} \cup E_e(H) \cup E(G_1)$, where $V_e(H) = \cup_{i=1}^{|E(G_1)|}V_i(H)$, $V_i(H) = V(H)\ \forall i$, $E_e(H) = \cup_{i=1}^{|E(G_1)|}E_i(H)$, $E_i(H) = E(H)\ \forall i$.

(3) $Q_H(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 with a copy of H and making every vertex in the ith copy of H adjacent to the end vertices of e_i for each $e_i \in E(G_1)$ along with edges joining all the vertices in the ith copy of H to all the vertices in the jth copy of H whenever e_i adjacent to e_j in G_1. That is, $Q_H(G_1)$ is a graph with vertex set $V(Q_H(G_1)) = V(G_1)\bigcup V_e(H)$ and edge set $E(Q_H(G_1)) = \{(v,h),(u,h) : e = vu \in E(G_1), h \in V_e(H)\} \cup E_e(H) \cup E(H_eVH_s)$ where $V_e(H) = \cup_{i=1}^{|E(G_1)|}V_i(H)$, $V_i(H) = V(H)\ \forall i$, $E(H_eVH_s) = \{(h_e,h_s) : h_e \in V(H_e), h_s \in V(H_s)\}$,$E_e(H) = \cup_{i=1}^{|E(G_1)|}E_i(H)$, $E_i(H) = E(H)\ \forall i$ and H_e, H_s are the copies of H corresponding to the edge $e, s \in E(G_1)$ and e, s are adjacent in G_1.

(4) $T_H(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 with a copy of H and making every vertex in the ith copy of H adjacent to the end vertices of e_i for each $e_i \in E(G_1)$ along with edges joining all the vertices in the ith copy of H to all the vertices in the jth copy of H

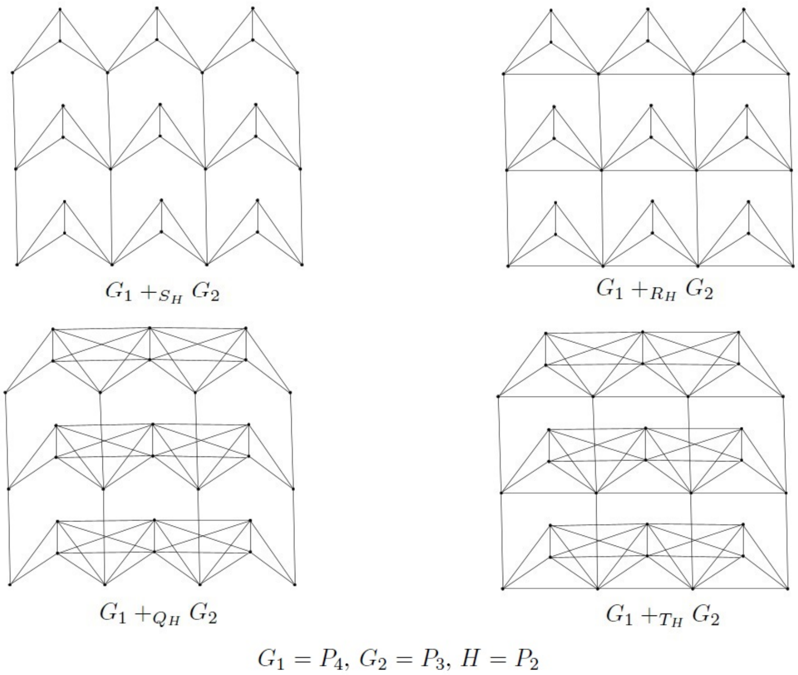

FIGURE 1.

whenever e_i adjacent to e_j in G_1 and keeping every edge of G_1 as well . That is, $T_H(G_1)$ is a graph with vertex set $V(T_H(G_1)) = V(G_1) \bigcup V_e(H)$ and edge set $E(T_H(G_1)) = E(G_1) \cup \{(v,h),(u,h) : e = vu \in E(G_1), h \in V(H)\} \cup E_e(H) \cup E(G_1) \cup E(H_eVH_s)$ where $V_e(H) = \cup_{i=1}^{|E(G_1)|} V_i(H)$, $V_i(H) = V(H)\ \forall i$, $E(H_eVH_s) = \{(h_e, h_s) : h_e \in V(H_e), h_s \in V(H_s)\}$, $E_e(H) = \cup_{i=1}^{|E(G_1)|} E_i(H)$, $E_i(H) = E(H)\ \forall i$ and H_e, H_s are the copies of H corresponding to the edge $e, s \in E(G_1)$ and e, s are adjacent in G_1. $T_H(G_1)$ is called the total graph associated with H.

Corresponding to the four new graphs $S_H(G_1), R_H(G_1), Q_H(G_1), T_H(G_1)$, we define four new sums called F_H sums associated with the graph H. Let F_H be any one of the symbols S_H, R_H, Q_H, T_H. The F_H sum of G_1 and G_2 is denoted by $G_1 +_{F_H} G_2$, is a graph with vertex set $V(G_1 +_{F_H} G_2) = V(F_H(G_1)) \times V(G_2)$ and the edge set $E(G_1 +_{F_H} G_2) = \{(a,b)(c,d) : a = c \in V(G_1)\ and\ bd \in E(G_2)\ or\ ac \in E(F_H(G_1))\ and\ b = d \in V(G_2)\}$. We consider the newly added vertices as white vertices and already existing vertices as black vertices. Let

$V(H) = \{u_1, u_2, \ldots, u_p\}$ in this discussion. By e_{ij} we mean the vertex u_j in the ith copy of H corresponding to the edge $e_i \in E(G_1)$. Figure 1 is an illustration with $G_1 = P_4, G_2 = P_3$ and $H = P_2$

3. Distance in $F_H(G)$, $F_H(G) = \{S_H(G), R_H(G), Q_H(G), T_H(G)\}$

Let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$ and let H be any graph with $V(H) = \{u_1, u_2, \ldots, u_p\}$. In each $S_H(G)$, $R_H(G)$, $Q_H(G)$, $T_H(G)$, the vertex e_{ij} is as defined above. The edge connecting u, v on a path is denoted by $u \to v$.

Lemma 3.1. *Let G be a connected graph and H be any graph, $u, v \in V(G)$. Then*

$$d_G(u,v) = \frac{d_{S_H(G)}(u,v)}{2} = d_{R_H(G)}(u,v) = d_{T_H(G)}(u,v) = \big(d_{Q_H(G)}(u,v)\big) - 1.$$

Proof. Let the vertices u, v in G are connected by a shortest path of $P : u = v_0 \to^{e_1} v_1 \to^{e_2} v_2 \ldots \to^{e_{t-1}} v_{t-1} \to^{e_t} v_t = v$ length t. Now fix a vertex $u_j \in V(H)$. Then in,

$\mathbf{S_H(G)}$: Since each edge in G is replaced by a graph H in $S_H(G)$ the edge $v_{i-1} \to^{e_i} v_i$ in P can be replaced by the shortest path $v_{i-1} \to e_{ij} \to v_i$ of length 2. Thus, every edge in P is replaced by a path of length 2 in $S_H(G)$. The new shortest path connecting u, v in $S_H(G)$ is

$$P_{S_H} : u = v_0 \to e_{1j} \to v_1 \to e_{2j} \to v_2 \ldots \to e_{t-1j} \to v_{t-1} \to e_{tj} v_t = v$$

$\mathbf{R_H(G)}, \mathbf{T_H(G)}$: The path P itself is a $u-v$ path in $R_H(G), T_H(G)$. Now we show that P is the shortest path, For let P^* be the shortest $u-v$ path in $R_H(G)$, if P^* consist only the vertices $v \in V(G)$, then $P^* = P$. If not, let e_i be a vertex other than $v \in V(G)$, then the $u_i \to e_i \to v_i$ can be replaced by the edges $u_i v_i$, thus P^* is not the shortest in $R_H(G)$. Similarly, Let P^* be the shortest path in $T_H(G)$, if P^* consist only the vertices of G_1 then $P = P^*$, if not there exist a section of P^* of the form $u_i \to e_i \to e_{i+1} \to \ldots \to e_j \to u_j$ of length $l+1$ with all the internal vertices of this section is in $V_e(H)$. Then, this section of P^* can be replaced by smaller section $u_i \to u_{i+1} \to \ldots u_{j-1} \to u_j$ of length l. Thus P^* is not the shortest, so P is the shortest path in $R_H(G), T_H(G)$.

$\mathbf{Q_H(G)}$: In $Q_H(G)$ to go from one vertex $u \in V(G)$ to another vertex $v \in V(G)$ we have to essentially go through an edge in G. Also the path

$e_{ij} \to v_i \to e_{(i+1)j}$ can be replaced by a shorter path $e_{ij} \to e_{(i+1)j}$. Thus a path P of length l in G

$$P : v_0 \to^{e_1} v_1 \to^{e_2} \ldots \to^{e_{l-1}} v_{l-1}$$

corresponds to a path of length $l+1$ in $Q_H(G_1)$ of the form

$$P_{Q_H(G)} : v_0 \to e_{1j} \to e_{2j} \ldots e_{(l-1)j} \to e_{(l-1)j}$$

□

Lemma 3.2. *Let G be a connected graph and H be any graph with $e_{ij}, e_{ik} \in V(H) \cap V(F_H(G))$ where $F_H = S_H$ or R_H or Q_H or T_H (vertices belonging to the same component of H). Then*

$$d_{F_H}(e_{ij}, e_{ik}) = \begin{cases} 1, & if\ (v_j, v_k) \in E(H) \\ 2, & otherwise \end{cases}.$$

Proof. For each path other than the edge $(e_{ij}, e_{ik}) \in E(F_H(G))$, we can replace the path by a shorter path $e_{ij} \to v_i \to e_{ik}$ of length 2. □

Lemma 3.3. *Let G be a connected graph and H be any other graph with $e_{ij}, e_{tk} \in V(H) \cap V(F_H(G))$ where $F_H = S_H$ or R_H or Q_H or T_H (vertices belonging to the different component of H). Then*

$$\frac{d_{S_H(G)}(e_{ij}, e_{tk})}{2} = d_{R_H(G)}(e_{ij}, e_{tk}) - 1 = d_{Q_H(G)}(e_{ij}, e_{tk}) = d_{T_H(G)}(e_{ij}, e_{tk})$$

Proof. Fix a vertex $u_j \in V(H)$, Let $e, f \in E(G)$ and the shortest path from e to f of length l in the line graph $L(G)$ be

$$e = e_0 \to^{v_1} e_1 \to^{v_2} e_3 \ldots \to^{v_l} e_l = f.$$

Then the corresponding shortest path of length l in $Q_H(G), T_H(G)$ is

$$e_{0j} \to^{v_1} e_{1j} \to^{v_2} e_{3j} \ldots \to^{v_l} e_{lj}.$$

From this path, we obtain a shortest path of length $2l$ in $S_H(G)$ as

$$e_{0j} \to v_1 \to e_{1j} \to v_2 \to e_{3j} \ldots \to v_l \to e_{lj}.$$

Similarly, from the above path, we obtain a shortest path of length $l+1$ in $R_H(G)$ as

$$e_{0j} \to v_1 \to v_2 \ldots \to v_l \to e_{lj}.$$

□

Lemma 3.4. *Let G be a connected graph and H be any other graph with $v \in V(G)$, $e_{ij} \in V(H) \cap V(F_H(G))$ where $F_H = S_H$ or R_H or Q_H or T_H. Then*

$$\frac{d_{S_H(G)}(v, e_{ij}) + 1}{2} = d_{R_H(G)}(v, e_{ij}) - 1 = d_{Q_H(G)}(v, e_{ij}) = d_{T_H(G)}(v, e_{ij}).$$

Proof. Consider a shortest path of length $2l - 1$ in $S_H(G)$ as

$$e_{0j} \to v_1 \to e_{1j} \to v_2 \to e_{3j} \ldots \to v_l.$$

From this, we obtain the shortest path of length l in $R_H(G)$ as

$$e_{0j} \to v_1 \to v_2 \ldots \to v_l.$$

The corresponding shortest path of length l in $Q_H(G)$ is

$$e_{0j} \to^{v_1} e_{1j} \to^{v_2} e_{3j} \ldots \to v_l.$$

We choose either one among the above two paths as the shortest path of length l in $T_H(G)$. $\square$

From this lemmas, we obtain the relationship of wiener index among the four graphs as

Theorem 3.1. *Let G be a connected graph and H be any graph with vertex sets $V(G)$, $V(H)$ and edge set $E(G)$, $E(H)$ respectively. Then*

(1) $W(S_H(G)) = 2W(T_H(G)) - |V(G)||E(G)||V(H)|$,
(2) $W(S_H(G)) = 2W(Q_H(G)) - |V(G)|(|V(G)| - 1) - |V(G)||E(G)||V(H)|$,
(3) $W(S_H(G)) = 2W(R_H(G)) - 2\left(|E(G)| - 1\right)|V(H)|^2 - |V(G)||E(G)||V(H)|$.

Proof. We divide the sum into three parts as

$$\sum_{u,v \in V(F_H(G))} d_{F_H(G)}(u, v) = \sum_{u,v \in V(G)} d_{F_H(G)}(u, v) + \sum_{e_{ij}, e_{kt} \in V(F_H(G)) \cap V(H)} d_{F_H(G)}(e_{ij}, e_{kt})$$
$$+ \sum_{u \in V(G) e_{ij} \in V(F_H(G)) \cap V(H)} d_{F_H(G)}(u, e_{ij})$$

and by using Lemmas 1-4 we get the desired result. $\square$

4. Polynomials associated with $F_H(G)$

The Wiener index of four graphs $F_H(G) = \{S_H(G), R_H(G), Q_H(G), T_H(G)\}$ are mutually related. A similar observation can be done in the case of Hosoya (Wiener) polynomials as well. Throughout this section we denote $W(G;p)$ as the Hosoya (Wiener) polynomial of G in the variable p.

Theorem 4.1. *Let G be a connected graph and H be any graph. Define*

$$A = \{e \in E(T_H(G)) : e = (u,v), u \text{ or } v \in V(G)\}.$$

Then

$$W(S_H(G);p) = \frac{W(T_H(G);p^2)}{p} + \left(1 - \frac{1}{p}\right) W(G;p^2) + \left(1 - \frac{1}{p}\right) W(T_H(G)/A;p^2).$$

Proof. Splitting the sum into three different parts,

$$\begin{aligned} W(S_H(G);p) &= \sum_{u,v\in V(G)} p^{d_{S_H(G)}(u,v)} + \sum_{e_{ij},e_{kt}\in V(F_H(G))\cap V(H)} p^{d_{S_H(G)}(e_{ij},e_{kt})} \\ &+ \sum_{u\in V(G) e_{ij}\in V(F_H(G))\cap V(H)} p^{d_{S_H(G)}(u,e_{ij})} \\ &= \sum_{u,v\in V(G)} p^{2d_{T_H(G)}(u,v)-1} + \sum_{e_{ij},e_{kt}\in V(F_H(G))\cap V(H)} p^{2d_{T_H(G)}(e_{ij},e_{kt})-1} \\ &+ \sum_{u\in V(G) e_{ij}\in V(F_H(G))\cap V(H)} p^{2d_{T_H(G)}(u,e_{ij})-1} + \sum_{u,v\in V(G)} p^{2d_G(u,v)}\left(1-\frac{1}{p}\right) \\ &+ \sum_{e_{ij},e_{kt}\in V(S_H(G))\cap V(H)} p^{2d_{T_H(G)}(e_{ij},e_{kt})}\left(1-\frac{1}{p}\right) \end{aligned}$$

$$W(S_H(G),p) = \frac{W(T_H(G);p^2)}{p} + \left(1 - \frac{1}{p}\right) W(G;p^2) + \left(1 - \frac{1}{p}\right) W(T_H(G)/A;p^2).$$

□

Theorem 4.2. *Let G be a connected graph and H be any graph. Define*

$$A = \{e \in E(T_H(G)) : e = (u,v), u \text{ or } v \in V(G)\}.$$

Then

$$W(R_H(G),p) = W(T_H(G);p) + (p-1)\, W(T_H(G)/A;p).$$

Proof. Splitting the sum into three different parts,

$$
\begin{aligned}
&W(R_H(G), p) \\
&= \sum_{u,v \in V(G)} p^{d_{R_H(G)}(u,v)} + \sum_{e_{ij}, e_{kt} \in V(R_H(G)) \cap V(H)} p^{d_{R_H(G)}(e_{ij}, e_{kt})} \\
&+ \sum_{u \in V(G) e_{ij} \in V(R_H(G)) \cap V(H)} p^{d_{R_H(G)}(u, e_{ij})} \\
&= \sum_{u,v \in V(G)} p^{d_{T_H(G)}(u,v)} + \sum_{e_{ij}, e_{kt} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(e_{ij}, e_{kt})} \\
&+ \sum_{u \in V(G) e_{ij} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(u, e_{ij})} + (p-1) \sum_{e_{ij}, e_{kt} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(e_{ij}, e_{kt})} \\
&= W(T_H(G); p) + (p-1)\, W(T_H(G)/A; p).
\end{aligned}
$$

□

Theorem 4.3. *Let G be a connected graph and H be any graph. Then*

$$W(Q_H(G), p) = W(T_H(G); p) + (p-1)\, W(G; p).$$

Proof. Splitting the sum into three different parts,

$$
\begin{aligned}
W(Q_H(G), p) &= \sum_{u,v \in V(G)} p^{d_{Q_H(G)}(u,v)} + \sum_{e_{ij}, e_{kt} \in V(Q_H(G)) \cap V(H)} p^{d_{Q_H(G)}(e_{ij}, e_{kt})} \\
&+ \sum_{u \in V(G) e_{ij} \in V(Q_H(G)) \cap V(H)} p^{d_{Q_H(G)}(u, e_{ij})} \\
&= \sum_{u,v \in V(G)} p^{d_{T_H(G)}(u,v)+1} + \sum_{e_{ij}, e_{kt} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(e_{ij}, e_{kt})} \\
&+ \sum_{u \in V(G) e_{ij} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(u, e_{ij})} \\
&= \sum_{u,v \in V(G)} p^{d_{T_H(G)}(u,v)} + \sum_{e_{ij}, e_{kt} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(e_{ij}, e_{kt})} \\
&+ \sum_{u \in V(G) e_{ij} \in V(T_H(G)) \cap V(H)} p^{d_{T_H(G)}(u, e_{ij})} + (p-1) \sum_{u,v \in V(G)} p^{d_{(G)}(u,v)} \\
&= W(T_H(G); p) + (p-1)\, W(G; p).
\end{aligned}
$$

□

5. The Wiener Index of F_H sums

In this section, we compute the Wiener index of the four F_H sums of graphs. Wiener index of the F_H sums can be computed by finding the distance relations among all kinds of vertices in the sum. So we first obtain the distances between various kinds vertices in $F_H = \{S_H, R_H, Q_H, T_H\}$.

Lemma 5.1.

a. *Let G_1 G_2 be two connected graphs and H be any graph $u = (u_1, u_2)$ be any black vertex. Then for all $v = (v_1, v_2) \in V(G_1 +_{F_H} G_2)$ with $F_H = S_H, R_H$ we have*

$$d(u, v|G_1 +_{F_H} G_2) = d(u_1, v_1|F_H(G_1)) + d(u_2, v_2|G_2).$$

b. *Let G_1 G_2 be two connected graphs and H be any graph. If $e_i, e_j \in E(G_1)$ then for all $u = (e_{ik}, u_2)$, $v = (e_{jt}, v_2) \in V(G_1 +_{F_H} G_2)$ with $u_2 \neq v_2$ $F_H = S_H, R_H$. Then:*

$$d(u, v|G_1 +_{F_H} G_2) = \begin{cases} 2 + d(u_2, v_2|G_2) & if\ i = j \\ d(e_{ik}, e_{jt}|F_H(G_1)) + d(u_2, v_2|G_2) & if\ i \neq j \end{cases}.$$

c. *Let G_1 G_2 be two connected graphs and H be any graph, If $e_i, e_j \in E(G_1)$ then for all $u = (e_{ik}, u_2)$, $v = (e_{jt}, v_2) \in V(G_1 +_{F_H} G_2)$ with $u_2 = v_2$ $F_H = S_H, R_H$,(white vertices in the same copy). Then:*

$$d(u, v|G_1 +_{F_H} G_2) = d(e_{ik}, e_{jt}|F_H(G_1)).$$

Proof.

a. Let $x = d(u, v|G_1 +_{F_H} G_2)$, $x_1 = d(u_1, v_1|F_H(G_1))$ and $x_2 = d(u_2, v_2|G_2)$, and let

$$P_1 : u_1 = t_0^1 \to t_1^1 \to \dots t_{x_1-1}^1 \to t_{x_1}^1 = v_1,$$
$$P_2 : u_2 = s_0^2 \to s_1^2 \to \dots s_{x_2-1}^2 \to s_{x_2}^2 = v_2,$$

be the corresponding shortest paths. Then using these two paths P_1, P_2 we easily construct a new path from u to v in $G_1 +_{F_H} G_2$ as

$$\begin{aligned} P_3 : (u_1, u_2) = (t_0^1, s_0^2) \to (t_1^1, s_0^2) \to \dots (t_{x_1-1}^1, s_0^2) \\ \to (t_{x_1}^1, s_0^2) = (v_1, s_0^2) \\ (v_1, s_0^2) \to (v_1, s_1^2) \to \dots (v_1, s_{x_2}^2) = (v_1, v_2). \end{aligned}$$

Thus, $x \leq x_1 + x_2$. Also corresponding to every path P from u to v in $G_1 +_{F_H} G_2$,

$$P : (u_1, u_2) = (t_0^1, s_0^2) \to (t_1^1, s_1^2) \to \dots (t_{x-1}^1, s_{x-1}^2) \\ \to (t_x^1, s_x^2) = (v_1, v_2),$$

we construct a path from u_1 to v_1 as $u_1 = t_0^1 \to t_1^1 \to \dots t_{y_1-1}^1 \to t_{y_1}^1 = v_1$ in $F_H(G_1)$ and a path of the form $u_2 = s_0^2 \to s_1^2 \to \dots s_{y_2-1}^2 \to s_{y_2}^2 = v_2$ from u_2 to v_2 in G_2 by replacing every consecutive similar vertex $(uuuu..u)$ by a single vertex(u). Thus $x_1 + x_2 \leq x = y_1 + y_2$, that is

$$d(u, v|G_1 +_{F_H} G_2) = d(u_1, v_1|F_H(G_1)) + d(u_2, v_2|G_2).$$

b. Consider the case $u = (e_{ik}, u_2)$, $v = (e_{jt}, v_2)$ with $u_2 \neq v_2$.

Case I $i = j$: Let $e_i = u_i v_i$ and $u_2 = s_0^2 \to s_1^2 \to \dots s_{y_2-1}^2 \to s_{y_2}^2 = v_2$ be the shortest path of length y_2 from u_2 to v_2 in G_2. Fix u_i, then we easily construct a shortest path from u to v using the edges $e_{ik}u_i$ and $e_{jt}u_i$ and the path from u_2 to v_2 as

$$P : (e_{ik}, u_2) = (e_{ik}, s_0^2) \to (u_i, s_0^2) \to (u_i, s_1^2) \dots (u_i, s_{y_2}^2) \\ \to (e_{jt}, s_{y_2}^2) = (e_{jt}, v_2)$$

of length $2 + d(u_2, v_2/G_2)$.

Case II $i \neq j$: Let $d(e_{ik}, e_{jt}|F_H(G_1)) = y_1$,$x = d(u, v|G_1 +_{F_H} G_2)$, $d(u_2, v_2) = x_1$ and let $e_{ik} = p_0^1 \to p_1^1 \to \dots p_{y_1-1}^1 \to p_{y_1}^1 = e_{jt}$ (where $p_{y_1-1}^1 \in V(G_1)$ since $F_H = S_H, R_H$) and $u_2 = s_0^2 \to s_1^2 \to \dots s_{x_1-1}^2 \to s_{x_1}^2 = v_2$ be the corresponding paths. Now we construct the following two paths in $G_1 +_{F_H} G_2$.

$$P_1 : u = (p_0^1, u_2) \to (p_1^1, u_2) \to \dots (p_{y_1-1}^1, u_2) = (p_{y_1-1}^1, s_0^2)$$
$$(p_{y_1-1}^1, s_0^2) \to (p_{y_1-1}^1, s_1^2) \to \dots (p_{y_1-1}^1, s_{x_1}^2) \to (e_{jt}, v_2) = v$$

of lengths $x_1 + y_1$, thus $x = d(u, v|G_1 +_{F_H} G_2) \leq x_1 + y_1$. to prove the reverse part, we assume that there exist a path P from u to v in $G_1 +_{F_H} G_2$ and proceeding as in the proof of (a.) we will establish that $x_1 + y_1 \leq x$. Thus, $d(u, v|G_1 +_{F_H} G_2) = d(e_{ik}, e_{jt}|F_H(G_1)) + d(u_2, v_2|G_2)$.

c. As in the proof of (a.) consider the shortest path form e_{ik} to e_{jt} of length y_1, $e_{ik} = p_0^1 \to p_1^1 \to \dots p_{y_1-1}^1 \to p_{y_1}^1 = e_{jt}$ which corresponds to a path $(e_{ik}, u_2) = (p_0^1, u_2) \to (p_1^1, u_2) \to \dots (p_{y_1-1}^1, u_2) \to (p_{y_1}^1, u_2) = (e_{jt}, u_2)$ of same length from u to v.Conversely, since $u_2 = v_2, d(u_2, v_2) = 0$ every such path P in $G_1 +_{F_H} G_2$ must have same second component thus the shortest path from u to v must be of the same length as the shortest path from e_{ik} to e_{jt}.

□

Lemma 5.2.

a. *Let G_1, G_2 be two connected graphs H be any graph and $u = (u_1, u_2)$ be any u black vertex. Then for all $v = (v_1, v_2) \in V(G_1 +_{F_H} G_2)$ with $F_H = Q_H, T_H$ we have*

$$d(u, v|G_1 +_{F_H} G_2) = d(u_1, v_1|F_H(G_1)) + d(u_2, v_2|G_2).$$

b. *Let G_1, G_2 be two connected graphs and H be any graph. If $e_i, e_j \in E(G_1)$, then for all $u = (e_{ik}, u_2)$, $v = (e_{jt}, v_2) \in V(G_1 +_{F_H} G_2)$ with $u_2 \neq v_2$ $F_H = Q_H, T_H$. Then:*

$$d(u, v|G_1 +_{F_H} G_2) =$$
$$\begin{cases} 2 + d(u_2, v_2|G_2) & if\ i = j\ u_2 \neq v_2 \\ 1 + d(e_{ik}, e_{jt}|F_H(G_1)) + d(u_2, v_2|G_2) & if\ i \neq j, u_2 \neq v_2 \end{cases}.$$

c. *Let G_1, G_2 be two connected graphs and H be any graph. If $e_i, e_j \in E(G_1)$, then for all $u = (e_{ik}, u_2)$, $v = (e_{jt}, v_2) \in V(G_1 +_{F_H} G_2)$ with $u_2 = v_2$ $F_H = Q_H, T_H$, (white vertices in the same copy). Then:*

$$d(u, v|G_1 +_{F_H} G_2) = d(e_{ik}, e_{jt}|F_H(G_1)).$$

Proof.

a. Proceed as in the case (a.) of Lemma 5, we easily obtain the result using similar arguments.

b. *Case I,* $i = j, u_2 \neq v_2$: Then proceed as in the Case I of Lemma 5(b) to get the results.
Case II $i \neq j, u_2 \neq v_2$: Let $d(e_{ik}, e_{jt}|F_H(G_1)) = y_1$, $d(u_2, v_2) = x_1$, $x = d(u, v|G_1 +_{F_H} G_2)$ and let $e_{ik} = p_0^1 \to p_1^1 \to \dots p_{y_1-1}^1 \to p_{y_1}^1 = e_{jt}$ and $u_2 = s_0^2 \to s_1^2 \to \dots s_{x_1-1}^2 \to s_{x_1}^2 = v_2$ be the corresponding paths. Let $u_i^1 v_i^1 = e_i$ and $u_j^1 v_j^1 = e_j$, also use the fact that the shortest path connecting e_{ik}, e_{jt} in $F_H = Q_H, T_H$ must be a path consisting only of vertices in the copies H $(V_e(H))$ as every path of $e_{ik} \to v_i^1 \to e_{jt}$ (where v_i^1 is common vertex of both the edges) can be replaced by a single

edges $e_{ik}e_{jt}$ which is shorter. Fix a vertex u_j^1 such that u_j^1 is closer to e_{ik} than v_j^1(we can similarly fix u_i^1 as well). Now we construct the following path P_1 in $G_1 +_{F_H} G_2$.

$$\begin{aligned}P_1 : u = (p_0^1, u_2) &\to (p_1^1, u_2) \to \ldots (p_{y_1-1}, u_2) \to (u_j^1, u_2)\\ &\to (u_j^1, s_1^2) \to \ldots (u_j^1, s_{x_1}^2) \to (e_{jt}, v_2) = v\end{aligned}$$

of lengths $1 + x_1 + y_1$, thus $x = d(u, v|G_1 +_{F_H} G_2) \leq 1 + x_1 + y_1$. To prove the reverse inequality, fix a vertex v_i^1 such that distance between v_i^1 and e_{jt} is the shortest compared to that from u_i^1 to e_{jt} in $F_H(G_1)$. Then (v_i^1, u_2) is a black vertex and let P be the shortest path between (e_{ik}, u_2) and (e_{jt}, v_2) of length x. Then we obtain a shortest path P^* from (v_i^1, u_2) to (e_{jt}, v_2) by deleting the edge connecting (e_{ik}, u_2) and (v_i^1, u_2) of length $x - 1$ in P. By replacing consecutive similar vertices in the first and second components of P^* by a single vertex, we get a path P_1 from v_i^1 to e_{jt} of length say s_1 and a path say P_2 from u_2 to v_2 of length s_2 in $F(G_1)$ and G_2 respectively. Since $d(v_i^1, e_{jt}) = d(e_{ik}, e_{jt})$ in $F(G_1)$, we have $s_1 + s_2 \geq x_1 + y_1$ so $x = 1 + s_1 + s_2 \geq 1 + x_1 + y_1$. Thus $x = d(u, v/G_1 +_{F_H} G_2) = 1 + x_1 + y_1 = 1 + d(e_{ik}, e_{jt}|F_H(G_1)) + d(u_2, v_2|G_2)$.

c. Proceed as in case (c.) of Lemma 5 to obtain the required result.

□

Using this, we find the Wiener index of F_H Sums in terms of the Wiener index of its component graphs.

Theorem 5.1. *Let G_1 , G_2 be two connected graphs and H be any graph and $F_H = S_H$ or R_H. Then*

$$\begin{aligned}W(G_1 +_{F_H} G_2) = |V(G_2)|^2 W(F_H(G_1)) &+ \left(|V(G_1)|^2 + (|E(G_1)||V(H)|)^2\right.\\ &\left.+2|V(G_1)||E(G_1)||V(H)|\right) W(G_2)\\ &+ \left(|V(G_2)|^2 - |V(G_2)|\right)(|E(G_1)||V(H)|).\end{aligned}$$

Proof. We divide the vertex set of $G_1 +_{F_H} G_2$ into two different subsets as

$$A = \{u = (u_1, v_1) \in V(G_1 +_{F_H} G_2) : u = (u_1, v_1) \in V(G_1) \times V(G_2)\},$$
$$B = \{u = (u_1, v_1) \in V(G_1 +_{F_H} G_2) : u_1 \in V_e(H), v_1 \in V(G_2)\}.$$

We divide the sum of the distances between the vertices of $G_1 +_{F_H} G_2$ into different components to calculate the Wiener index:

$$\begin{aligned} W(G_1 +_{F_H} G_2) &= \frac{1}{2}\sum_{u,v\in A} d(u,v|G_1 +_{F_H} G_2) + \frac{1}{2}\sum_{u\in A, v\in B} d(u,v|G_1 +_{F_H} G_2) \\ &\quad + \frac{1}{2}\sum_{u,v\in B} d(u,v|G_1 +_{F_H} G_2). \end{aligned}$$

Now we find the three sums separately. To find the first sum, we use the fact that $\forall\, u = (u_1, v_1), v = (u_2, v_2) \in A$, $d(u, v|G_1 +_{F_H} G_2) = d((u_1, v_1), (u_2, v_2)|G_1 +_{F_H} G_2)$ $d((u_1, v_1), (u_2, v_2)|G_1 +_{F_H} G_2) = d(u_1, u_2|F_H(G_1)) + d(v_1, v_2|G_2)$ and Lemma 5.

$$\begin{aligned} &A_1 \\ &= \frac{1}{2}\sum_{u,v\in A} d(u,v|G_1 +_{F_H} G_2) = \frac{1}{2}\sum_{u,v\in A} d((u_1,v_1),(u_2,v_2)|G_1 +_{F_H} G_2) \\ &= \frac{1}{2}\sum_{(u_1,v_1),(u_2,v_2)} d(u_1,u_2|F_H(G_1)) + d(v_1,v_2|G_2) \\ &= \frac{1}{2}\left(\sum_{u_1,u_2\in V(G_1)}\ \sum_{v_1,v_2\in V(G_2)} d(u_1,u_2|F_H(G_1)) + \sum_{u_1,u_2\in V(G_1)}\ \sum_{v_1,v_2\in V(G_2)} d(v_1,v_2|G_2)\right) \\ &= \frac{1}{2}|V(G_2)|^2 \sum_{u_1,u_2\in V(G_1)} d(u_1,u_2|F_H(G_1)) + |V(G_1)|^2 W(G_2). \end{aligned}$$

Now consider the case where $u = (u_1, v_1) \in V(G_1) \times V(G_2)$, $v = (e_{jt}, v_2)$, $e_{jt} \in V_e(H)$, $v_2 \in V(G_2)$ (or vice versa). By Lemma 5 we have $d(u, v|G_1 +_{F_H} G_2) = d((u_1, v_1)(e_{jt}, v_2)|G_1 +_{F_H} G_2) = d(u_1, e_{jt}|F_H(G_1)) + d(v_1, v_2|G_2)$,

$$\begin{aligned} &\frac{1}{2}\sum_{u\in A, v\in B} d(u,v|G_1 +_{F_H} G_2) \\ &= \frac{1}{2}\sum_{u\in A, v\in B} d((u_1,e_{jt}),(v_1,v_2)|G_1 +_{F_H} G_2) \\ &= \frac{1}{2}\sum_{(u_1,v_1),(e_{jt},v_2)} (d(u_1,e_{jt}|F_H(G_1)) + d(v_1,v_2|G_2)) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{u_1, \in V(G_1)} \sum_{e_{jt} \in V_e(H)} \sum_{v_1, v_2 \in V(G_2)} d(u_1, e_{jt} | F_H(G_1)) \\
&+ \frac{1}{2} \sum_{u_1 \in V(G_1)} \sum_{e_{jt} \in V_e(H)} \sum_{v_1, v_2 \in V(G_2)} d(v_1, v_2 | G_2) \\
&= \frac{1}{2} |V(G_2)|^2 \sum_{u_1 \in V(G_1)} \sum_{e_{jt} \in V_e(H)} d(u_1, e_{jt} | F_H(G_1)) \\
&+ |V(G_1)||E(G_1)||V(H)|W(G_2).
\end{aligned}
$$

By considering the reverse case as well, the total distance is

$$
\begin{aligned}
A_2 &= \frac{1}{2} \sum_{u \in A, v \in B} d(u, v | G_1 +_{F_H} G_2) \\
&= |V(G_2)|^2 \sum_{u_1 \in V(G_1)} \sum_{e_{jt} \in V_e(H)} d(u_1, e_{jt} | F_H(G_1)) \\
&+ 2|V(G_1)||E(G_1)||V(H)|W(G_2).
\end{aligned}
$$

Now consider the case where $u = (e_{ij}, v_1)$, $v = (e_{jt}, v_2), e_{ij}, e_{jt} \in V_e(H), v_1, v_2 \in V(G_2)$. We divide the sum into three different parts with respect to $i = j, v_1 \neq v_2$, $i \neq j, v_1 \neq v_2$ and $i = j, v_1 = v_2$. Let

$$
\begin{aligned}
S_1 &= \frac{1}{2} \sum_{u, v \in B} \{d(u, v | G_1 +_{F_H} G_2) : v_1 \neq v_2, i = j\}, \\
S_2 &= \frac{1}{2} \sum_{u, v \in B} \{d(u, v | G_1 +_{F_H} G_2) : v_1 \neq v_2, i \neq j\}, \\
S_3 &= \frac{1}{2} \sum_{u, v \in B} \{d(u, v | G_1 +_{F_H} G_2) : v_1 = v_2, i = j\}.
\end{aligned}
$$

Then,

$$
A_3 = \frac{1}{2} \sum_{u, v \in B} d(u, v | G_1 +_{F_H} G_2) = S_1 + S_2 + S_3. \tag{5.1}
$$

Now,

$$S_1 = \frac{1}{2} \sum_{(e_{ik},v_1)(e_{jt},v_2)\in B} 2 + d(v_1, v_2|G_2)$$

$$= \frac{1}{2} \sum_{v_1,v_2\in V(G_2) v_1\neq v_2} \sum_{e_{ik}e_{jt},\in V_e(H)\ i=j} 2$$

$$+ \frac{1}{2} \sum_{v_1,v_2\in V(G_2) v_1\neq v_2} \sum_{e_{ik}e_{jt},\in V_e(H)\ i=j} d(v_1, v_2|G_2)$$

$$= \left(|V(G_2)|^2 - |V(G_2)|\right) |E(G_1)||V(H)| + (|E(G_1)||V(H)|)\, W(G_2).$$

Similarly,

$$S_2 = \frac{1}{2} \sum_{u,v\in B} d(e_{ik}, e_{jt}|F_H(G_1)) + d(v_1, v_2|G_2)$$

$$= \frac{1}{2} \sum_{v_1,v_2\in V(G_2)} \sum_{e_{ik}e_{jt}\in V_e(H) i\neq j} d(e_{ik}, e_{jt}|F_H(G_1))$$

$$+ \frac{1}{2} \sum_{v_1,v_2\in V(G_2)} \sum_{e_{ik}e_{jt}\in V_e(H) i\neq j} d(v_1, v_2|G_2)$$

$$= \frac{1}{2} \sum_{v_1,v_2\in V(G_2)} \sum_{e_{ik}e_{jt}\in V_e(H) i\neq j} d(e_{ik}, e_{jt}|F_H(G_1))$$

$$+ \left((|E(G_1)||V(H)|)^2 - |E(G_1)||V(H)|\right) W(G_2).$$

Also,

$$S_3 = \frac{1}{2} \sum_{u,v\in B} \{d(u, v|G_1 +_{F_H} G_2) : v_1 = v_2, i = j\}$$

$$= \frac{1}{2} \sum_{v_1,v_2\in V(G_2) v_1=v_2} \sum_{e_{ik}e_{jt}\in V_e(H) i=j} d(e_{ik}, e_{jt}|F_H(G_1)),$$

$$W(G_1 +_{F_H} G_2) = A_1 + A_2 + A_3$$

$$= \frac{1}{2}|V(G_2)|^2 \sum_{u_1,u_2\in V(G_1)} d(u_1, u_2|F_H(G_1)) + |V(G_1)|^2 W(G_2)$$

$$+ |V(G_2)|^2 \sum_{u_1\in V(G_1)} \sum_{e_{jt}\in V_e(H)} d(u_1, e_{jt}|F_H(G_1)) + 2|V(G_1)||E(G_1)||V(H)|W(G_2)$$

$$
\begin{aligned}
&+ \sum_{v_1,v_2\in V(G_2) v_1\neq v_2} \sum_{e_{ik}e_{jt},\in V_e(H)\ i=j} 2 + \sum_{v_1,v_2\in V(G_2) v_1\neq v_2} \sum_{e_{ik}e_{jt},\in V_e(H)\ i=j} d(v_1,v_2|G_2)\\
&+ \frac{1}{2}|V(G_2)|^2 \sum_{e_{ik}e_{jt}\in V_e(H) i\neq j} d(e_{ik},e_{jt}|F_H(G_1))\\
&+ \left((|E(G_1)||V(H)|)^2 - |E(G_1)||V(H)|\right) W(G_2)\\
&+ \frac{1}{2}|V(G_2)|^2 \sum_{e_{ik},e_{jt}\in V_e(H) i=j} d(e_{ik},e_{jt}|F_H(G_1))\\
&= |V(G_2)|^2 W(F_H(G_1)) + \left(|V(G_1)|^2 + (|E(G_1)||V(H)|)^2\right.\\
&\left.+2|V(G_1)||E(G_1)||V(H)|\right) W(G_2)\\
&+ \left(|V(G_2)|^2 - |V(G_2)|\right)(|E(G_1)||V(H)|)
\end{aligned}
$$

□

Theorem 5.2. *Let G_1, G_2 be two connected graphs, H be any graph and $F_H = Q_H$ or T_H. Then*

$$
\begin{aligned}
&W(G_1 +_{F_H} G_2)\\
&= |V(G_2)|^2 W(F_H(G_1)) + \left(|V(G_1)|^2 + (|E(G_1)||V(H)|)^2\right.\\
&\left.+2|V(G_1)||E(G_1)||V(H)|\right) W(G_2) + \frac{1}{2}\left((|E(G_1)||V(H)|)^2\right.\\
&\left.+|E(G_1)||V(H)|\right)\left(|V(G_2)|^2 - |V(G_2)|\right)
\end{aligned}
$$

Proof. Define A, B as in Theorem 1. Then

$$
\begin{aligned}
W(G_1 +_{F_H} G_2) &= \frac{1}{2}\sum_{u,v\in A} d(u,v|G_1 +_{F_H} G_2) + \frac{1}{2}\sum_{u\in A, v\in B} d(u,v|G_1 +_{F_H} G_2)\\
&+ \frac{1}{2}\sum_{u,v\in B} d(u,v|G_1 +_{F_H} G_2).
\end{aligned}
$$

As in Theorem 1 we divide the sum into three different parts and calculate the sum of the distances. Now, the first two parts are as same as in Theorem 1, so its enough to find only the third sum. As in the previous cases we break down the third sum into four different parts

$$\sum_{u,v\in B} d(u,v|G_1 +_{F_H} G_2) = C_1 + C_2 + C_3 + C_4.$$

Let $u=(e_{ij},v_1)$, $v=(e_{jt},v_2), e_{ij}, e_{jt} \in V_e(H), v_1, v_2 \in V(G_2)$. Then,

$$C_1 = \frac{1}{2}\sum_{u,v\in B}\{d(u,v|G_1 +_{F_H} G_2) : v_1 \neq v_2, i=j\}$$

$$C_2 = \frac{1}{2}\sum_{u,v\in B}\{d(u,v|G_1 +_{F_H} G_2) : v_1 = v_2, i\neq j\}$$

$$C_3 = \frac{1}{2}\sum_{u,v\in B}\{d(u,v|G_1 +_{F_H} G_2) : v_1 = v_2, i=j\}$$

$$C_4 = \frac{1}{2}\sum_{u,v\in B}\{d(u,v|G_1 +_{F_H} G_2) : v_1 \neq v_2, i\neq j\}$$

By Lemma 6, we have,

$$\begin{aligned}
C_1 &= \frac{1}{2}\sum_{u,v\in B}\{d(u,v|G_1 +_{F_H} G_2) : v_1 \neq v_2, i=j\} \\
&= \frac{1}{2}\sum_{(e_{ik},v_1)(e_{jt},v_2)\in B} 2 + d(v_1,v_2|G_2) \\
&= \frac{1}{2}\sum_{v_1,v_2\in V(G_2) v_1\neq v_2}\ \sum_{e_{ik}e_{jt},\in V_e(H)\ i=j} 2 \\
&+ \frac{1}{2}\sum_{v_1,v_2\in V(G_2) v_1\neq v_2}\ \sum_{e_{ik}e_{jt},\in V_e(H)\ i=j} d(v_1,v_2|G_2) \\
&= \left(|V(G_2)|^2 - |V(G_2)|\right)|E(G_1)||V(H)| + \left(|E(G_1)||V(H)|\right) W(G_2
\end{aligned}$$

Now,

$$\begin{aligned}
C_2 &= \frac{1}{2}\sum_{u,v\in B}\{d(u,v|G_1 +_{F_H} G_2) : v_1 = v_2, i\neq j\} \\
&= \frac{1}{2}\sum_{v_1,v_2\in V(G_2) v_1= v_2}\ \sum_{e_{ik}e_{jt}\in V_e(H) i\neq j} d(e_{ik},e_{jt}|F_H(G_1)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
C_3 &= \frac{1}{2}\sum_{u,v\in B}\{d(u,v|G_1 +_{F_H} G_2) : v_1 = v_2, i=j\} \\
&= \frac{1}{2}\sum_{v_1,v_2\in V(G_2) v_1= v_2}\ \sum_{e_{ik}e_{jt}\in V_e(H) i=j} d(e_{ik},e_{jt}|F_H(G_1)).
\end{aligned}$$

Similarly, by Lemma 6,

$$
\begin{aligned}
C_4 &= \frac{1}{2}\sum_{u,v\in B}\{d(u,v|G_1 +_{F_H} G_2) : v_1 \neq v_2, i \neq j\}\\
&= \frac{1}{2}\sum_{u,v\in B}\left(1 + d(e_{ik}, e_{jt}|F_H(G_1)) + d(v_1, v_2|G_2)\right)\\
&= \frac{1}{2}\sum_{v_1,v_2\in V(G_2) v_1\neq v_2}\ \sum_{e_{ik}e_{jt}\in V_e(H) i\neq j} 1\\
&+ \frac{1}{2}\sum_{v_1,v_2\in V(G_2) v_1\neq v_2}\ \sum_{e_{ik}e_{jt}\in V_e(H) i\neq j} d(e_{ik}, e_{jt}|F_H(G_1))\\
&+ \frac{1}{2}\sum_{v_1,v_2\in V(G_2) v_1\neq v_2}\ \sum_{e_{ik}e_{jt}\in V_e(H) i\neq j} d(v_1, v_2|G_2)\\
&= \big((|E(G_1)||V(H)|)^2 - |E(G_1)||V(H)|\big)\big(|V(G_2)|^2 - |V(G_2)|\big)\\
&+ \frac{1}{2}\sum_{v_1,v_2\in V(G_2)}\ \sum_{e_{ik}e_{jt}\in V_e(H) i\neq j} d(e_{ik}, e_{jt}|F_H(G_1))\\
&+ \big((|E(G_1)||V(H)|)^2 - |E(G_1)||V(H)|\big)W(G_2).
\end{aligned}
$$

Thus we obtain,

$$
\begin{aligned}
&W(G_1 +_{F_H} G_2)\\
&= \frac{1}{2}|V(G_2)|^2\sum_{u_1,u_2\in V(G_1)} d(u_1, u_2|F_H(G_1)) + |V(G_1)|^2 W(G_2)\\
&+ |V(G_2)|^2\sum_{u_1\in V(G_1)}\ \sum_{e_{jt}\in V_e(H)} d(u_1, e_{jt}|F_H(G_1)) + 2|V(G_1)||E(G_1)||V(H)|W(G_2)\\
&+ \big(|V(G_2)|^2 - |V(G_2)|\big)|E(G_1)||V(H)| + (|E(G_1)||V(H)|)\,W(G_2)\\
&+ \frac{1}{2}\sum_{v_1,v_2\in V(G_2) v_1=v_2}\ \sum_{e_{ik}e_{jt}\in V_e(H) i\neq j} d(e_{ik}, e_{jt}|F_H(G_1))\\
&+ \frac{1}{2}\sum_{v_1,v_2\in V(G_2) v_1=v_2}\ \sum_{e_{ik}e_{jt}\in V_e(H) i=j} d(e_{ik}, e_{jt}|F_H(G_1))\\
&+ \frac{1}{2}\big((|E(G_1)||V(H)|)^2 - |E(G_1)||V(H)|\big)\big(|V(G_2)|^2 - |V(G_2)|\big)
\end{aligned}
$$

$$+\frac{1}{2}\sum_{v_1,v_2\in V(G_2)}\sum_{e_{ik}e_{jt}\in V_e(H)i\neq j} d(e_{ik},e_{jt}|F_H(G_1))$$
$$+\left((|E(G_1)||V(H)|)^2-|E(G_1)||V(H)|\right)W(G_2)$$
$$=|V(G_2)|^2W(F_H(G_1))+\left(|V(G_1)|^2+(|E(G_1)||V(H)|)^2\right.$$
$$\left.+2|V(G_1)||E(G_1)||V(H)|\right)W(G_2)$$
$$+\frac{1}{2}\left((|E(G_1)||V(H)|)^2+|E(G_1)||V(H)|\right)\left(|V(G_2)|^2-|V(G_2)|\right)$$

□

Illustration 1. *If $H=K_1$, we obtain the results in [5].*

Illustration 2. *If G_1, G_2, H are paths P_n, P_m, P_r with $n,m>3$ respectively, then the Wiener index is:*

a. $W(P_n +_{S_{P_r}} P_m) = \frac{m}{6}(2n^3r^2m + n^2r^2m^2 - 6n^2r^2m + 4n^3rm + 2n^2rm^2 - 2nr^2m^2 - 6n^2rm + 10nr^2m - 2nrm^2 + 2n^3m + n^2r^2 + r^2m^2 - 4nrm + 2n^2r - 2nr^2 + n^2m - 6mr^2 - 8nr + 6mr + 6mn + n^2 + r^2 - 6m + 6r)$;

b. $W(P_n +_{R_{P_r}} P_m) = \frac{m}{6}(n^3r^2m + n^2r^2m^2 + 2n^3rm + 2n^2rm^2 - 2nr^2m^2 - nr^2m + n^3m - 2nrm^2 + n^2m^2 - n^2r^2 + r^2m^2 - 8nrm - 2n^2r + 2nr^2 + 5nm - 4nr + 6rm - n^2 - r^2 - 6m + 6r)$;

c. $W(P_n +_{Q_{P_r}} P_m) = \frac{m}{6}(n^3r^2m + n^2r^2m^2 + 2n^3rm + 2n^2rm^2 - 2nr^2m^2 + 2nr^2m - 2nrm^2 + n^3m + n^2m^2 - 4n^2r^2 + r^2m^2 - 11nrm - 2n^2r + 8nr^2 - 3r^2m + 3n^2m + 2nm - nr + 9rm - n^2 - 4r^2 - 6m + 3r)$;

d. $W(P_n +_{T_{P_r}} P_m) = \frac{m}{6}(n^3r^2m + n^2r^2m^2 + 2n^3rm + 2n^2rm^2 - 2nr^2m^2 + 2nr^2m - 2nrm^2 + n^3m + n^2m^2 - 4n^2r^2 + r^2m^2 - 11nrm - 2n^2r + 8nr^2 - 3r^2m + 5nm - nr + 9rm - n^2 - 4r^2 - 6m + 3r)$.

6. Conclusions

In this paper we have defined F_H sums of graphs and obtained the relationship among distances of these four graphs and their Hosoya polynomial. We also computed the Wiener index of F_H sums of two connected graphs. These sums can be defined in terms of various other products such as strong product and lexicographic product. Other topological indices can also be computed for the F_H sums.

REFERENCES

[1] A.A. DOBRYNIN, R. ENTRINGER, I. GUTMAN: *Wiener index of trees: theory and applications*, Acta Appl. Math. **66** (2001), 211–249.

[2] D.M. CVETKOVIĆ, M. DOOB, H. SACHS: *Spectra of Graphs*, Academic Press, NewYprk, 1980.

[3] H. HOSOYA: *Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons*, Bull. Chem. Soc. Jpn. **4** (1971), 2332–2339.

[4] H. DENG, D. SARALA, S.K. AYYASWAMY, S. BALACHANDRAN: *The Zagreb indices of four operations on graphs*, Applied Mathematics and Computation, **275** (2016), 422-431.

[5] M. ELIASI, B. TAERI: *Four new sums of graphs and their Wiener indices*, Discrete Appl. Math. **157** (2009), 794–803.

[6] H. DENG, D. SARALA, S.K. AYYASWAMY, S. BALACHANDRAN: *The Zagreb indices of four operations on graphs*, Applied Mathematics and Computation, **275** (2016), 422-431.

[7] JIA- BAO LIU, MUHAMMAD JAVAID, HAFIZ MUHAMMAD AWAIS: *Computing Zagreb indices of the Subdivision Related Generalized Operations of Graphs*, IEEE Access, **7** (2019), 105479 - 105488.

[8] I. GUTMAN, K.C. DAS: *The first Zagreb index 30 years after*, MATCH Commun. Math. Comput. Chem. **50** (2004), 83–92.

[9] SHEHNAZ AKHTER,MUHAMMAD IMRAN: *Computing the Forgotten topological index of four operations on graphs*, AKCE International Journal of Graphs and Combinatorics, **14** (1) (2017), 70-79.

[10] D. STEVANOVIĆ: *Hosoya polynomial of composite graphs*, Discrete Math. **235** (2001), 237–244.

[11] H. WIENER: *Structural determination of paraffin boiling points*, J. Am. Chem. Soc. **69** (1947), 17–20.

[12] W. YAN, B.-Y. YANG, Y.-N. YEH: *The behavior of Wiener indices and polynomials of graphs under five graph decorations*, Applied Mathematics Letters **20** (2007), 290–295.

[13] Y.-N. YEH, I. GUTMAN: *On the sum of all distances in composite graphs*, Discrete Math. **135** (1994), 359–365.

DEPARTMENT OF MATHEMATICS, BISHOP CHULAPARAMBIL MEMORIAL COLLEGE, KOTTAYAM - 686001, KERALA, INDIA AND , MARTHOMA COLLEGE, THIRUVALLA, PATHANAMTHITTA - 689103, INDIA.

Email address: `lijualex0@gmail.com`

DEPARTMENT OF MATHEMATICS, ST.ALOYSIUS COLLEGE, EDATHUA, ALAPPUZHA - 689573, INDIA.

Email address: `indulalgopal@gmail.com`