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ON THE PROPERTIES OF BIFRACTIONAL BROWNIAN MOTION

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ABSTRACT. In this paper, we study several properties of the bifractional Brownian motion introduced by Houdré and Villa.

1. INTRODUCTION

In mathematics, a self-similar object is exactly or approximately similar to a part of it self.

Self-similarity is a major part of the mathematics.

On the one hand if you assume that you observe a self similar phenomens with no structure.

On the other hand if you want to have a complete classification of self similar fields then we can find in the literature a lot of counter-examples.

The self similarity and the stationarity of the increments are two main properties for which the fractional Brownian motion exhibited success as a modeling tool in engineering, mathematical finance, hydrology etc...

We will focus our attention to a gaussian process that generalize the fractional Brownian motion, called bifractional Brownian motion. Recall that the fBm is the only Self-similar Gaussian process with stationary increments starting from zero for small increments, in models such as turbulence, fBm seems a good model but inadequate for large increments. For this reason, in [29] the authors

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introduced an extension of the fMb keeping some properties (Self-similarity, gaussianity, stationarity for small increments) but enlarged the modeling tool kit.

We will pay a special attention to the case HK = $\frac{1}{2}$ (and K $\neq 1$, if K = 1 then H = $\frac{1}{2}$ and we have a Brownian motion).

In this case we will show that B^{H,K} admits a non-trivial quadratic variation equal to constant time t, this different from the fractional situation.

Let us summarize the results proved below

- Al though 2HK = 1 implies $H > \frac{1}{2}$, the process B^{H,K} seems in this case to have similar properties as the fBm $H < \frac{1}{2}$ it is short-memory.
- Nevertheless, having finite energy, it is also linked to the fBm with parameter bigger than $\frac{1}{2}$.

Russo and Tudor have established some properties on the strong variations, local times and stochastic calculus of real-valued bifractional Brownian motion.

An interesting property that deserves to be recalled in the fact, when $HK = \frac{1}{2}$, the quadratic variation of this process on $[0, t]$ is equal to a constant time t. This is really remarkable since as far as we know this is the only Gaussian Self - similar process with quadratic variation besides brownian motion.

Taking into acount this property, it is natural to ask if the bifractional Brownian motion B^{H,K} with HK = $\frac{1}{2}$ shares other properties with Brownian motion.

2. Preliminaries

Definition 2.1. *The bifractional brownian motion* $\left(B_t^{H,K}\right)_{t\geq 0}$ *is a centered Gaussian process, starting from zero, with covariance*

$$R^{H,K}(t,s) = \frac{1}{2^K}\left[(t^{2H} + s^{2H})^K - |t-s|^{2HK}\right],$$

with H $\in (0,1)$ *and K* $\in (0,1]$.

Note that, if K = 1 then $B^{H,1}$ is a fractional Brownian motion with Hurst parameter H $\in (0,1)$.

if $\sigma_\varepsilon^2(t) := E(B_{t+\varepsilon}^{H,K} - B_t^{H,K})^2)$, then $\lim\limits_{\varepsilon\to 0}\frac{\sigma_\varepsilon^2(t)}{\varepsilon^{2HK}} = 2^{1-K}$.

Let $T > 0$. For every s, t $\in$[0, T], we have

$$2^{-K}|t-s|^{2HK} \leq E(B_t^{H,K} - B_s^{H,K})^2 \leq 2^{-K}|t-s|^{2HK}.$$

Inequality shows that the process $B^{H,K}$ is a quasi-helix in the sense of J.P. Kahane for various properties and applications of quasi-helices).

For every $H \in (0,1)$ and K $\in (0,1)$,

$$\lim_{\varepsilon \to 0} \sup_{t \in [t_0-\varepsilon, t_0+\varepsilon]} |\frac{B_t^{H,K} - B_{t_0}^{H,K}}{t-t_0}| = +\infty,$$

with probability one for every t_0.

The process is HK-self-similar.

The process is Holder continuous of order δ for any $\delta < HK$. This follows from the Kolmogorov criterium.

3. Some properties of bifractional Brownian motion

Proposition 3.1. *Let X a bifractional Brownian motion with index HK then*

1. $E((X(t) - X(s))^2) \leq 2|t-s|^{2HK}$.
2. *X is locally Holder continuous for every exponent* $0 < r <$*HK.*
3. *At every point t the pointwise Holder exponent of X is HK.*

Proof.

1. In the beginnin

$$E(X(t) - X(s))^2 = 2^{1-K}|t-s|^{2HK} + (|t|^{2HK} + |s|^{2HK} - 2^{1-K}(|t|^{2H} + |s|^{2H})^K)$$

Since $(|t|^{2HK}+|s|^{2HK}-2^{1-K}(|t|^{2H}+|s|^{2H})^K)$ is non-negative by concavity:

$$E(X(t) - X(s))^2 \leq 2^{1-K}|t-s|^{2HK}$$

2. the theorem proves that X is a least HK locally Holder continuous.
3. It remains to prove that index HK is the best Holder exponent. Fix a point t. Let $\sigma_n^2 = E(X(t+\frac{1}{n}) - X(t))^2$.

   There exists a positive constant C such that $\sigma_n > Cn^{-HK}$ as $n \mapsto +\infty$. Then one can show that for $r >$HK,

$$\lim_{n \to +\infty} \frac{|\frac{1}{n}|^\gamma}{X(t+\frac{1}{n})X(t)} = 0(d).$$

The limit is also true in a convergence in probability sense. One can find a sequence

$$\frac{1}{n} \longmapsto 0 \quad \text{such that} \quad \lim_{n \to +\infty} \frac{|\frac{1}{n}|^\gamma}{X(t+\frac{1}{n})X(t)} = 0 \quad \text{a.s.}$$

This yields that

$$\lim_{n\to+\infty}\frac{X(t+\frac{1}{n})-X(t)}{|\frac{1}{n}|^{\gamma}}=+\infty \quad \text{a.s.}$$

and that the pointwise Holder exponent is lower than every $H^{'} > H$ almost surely.

One can conclude that this exponent is bigger than $H^{'} < H$ almost surely.

□

Proposition 3.2. *Let $(B_t^{H,K})_{t\in[0,T]}$ be a bifractional motion with parameters $H \in (0;1)$ and $K \in (0,1)$. Then*

$$\left[B^{H,K}\right]_t^{(K)}=0 \quad \textit{if} \quad \alpha>\frac{1}{HK} \quad \textit{and} \quad \left[B^{H,K}\right]_t^{(K)}=2^{\frac{1-K}{2HK}}\varphi HK^t \quad \textit{if} \quad \alpha=\frac{1}{HK},$$

where $\varphi_{HK}=E|N|^{\frac{1}{HK}}$. N being a standard normal random variable.

Proof. We define

$$C_\varepsilon^{(\alpha)}(t)=\frac{1}{\varepsilon}\int_0^t|B_{s+\varepsilon}^{H,K}-B_s^{H,K}|ds.$$

It suffices to show than $C_\varepsilon^{\frac{1}{HK}}(t)$ converges in $L^2(\Omega)$ as $\varepsilon\mapsto 0$ to $2^{\frac{1-K}{HK}}\varphi HK^t$. So,

$$E|B_{s+\varepsilon}^{H,K}-B_s^{H,K}|^{\frac{1}{HK}}=\Big(E|B_{s+\varepsilon}^{H,K}-B_s^{H,K}|^2\Big)^{\frac{1}{HK}}\simeq 2^{\frac{1-K}{HK}}\varphi HK^t,$$

and therefore

$$\lim_{\varepsilon\to 0}E(C_\varepsilon^{\frac{1}{HK}}(t))=2^{\frac{1-K}{HK}}\varphi HK^t.$$

To obtain the conclusion it suffices to show that

$$\lim_{\varepsilon\to 0}E(C_\varepsilon^{\frac{1}{HK}}(t))^2=(2^{\frac{1-K}{HK}}\varphi HK^t)^2t^2.$$

We have

$$E(C_\varepsilon^{\frac{1}{HK}}(t))^2=\frac{2}{\varepsilon^2}\int_0^t\int_0^u E|B_{u+\varepsilon}^{H,K}-B_u^{H,K}\cdot B_{v+\varepsilon}^{H,K}-B_v^{H,K}|^{\frac{1}{HK}},$$

$$E\left(|\ N_1\ |^{\frac{1}{HK}}\ |\frac{\Theta_\varepsilon(u,v)}{C_1\varepsilon^{2HK}}N_1+2^{1-K}N_2\sqrt{1-\left(\frac{\Theta_\varepsilon(u,v)}{C_2\varepsilon^{2HK}}\right)^2}|^{\frac{1}{HK}}\right),$$

where C_1, C_2 are strictly positive constants and we defined

$$\Theta_\varepsilon(u,v)=E(B_{u+\varepsilon}^{H,K}-B_u^{H,K})(B_{v+\varepsilon}^{H,K}-B_v^{H,K}).$$

We compute $\Theta_\varepsilon(u,v) = (a_\varepsilon(u,v) + b_\varepsilon(u,v))$ where

$$a_\varepsilon(u,v) = \frac{1}{2^K}[((u+\varepsilon)^{2H} + (v+\varepsilon)^{2H})^K - ((u+\varepsilon)^{2H}v^{2H})^K - ((v+\varepsilon)^{2H} + u^{2H})^K + (u^{2H}+v^{2H})^K]$$

and

$$b_\varepsilon(u,v) = \left[(u+\varepsilon-v)^{2HK} + (u-\varepsilon-v)^{2HK} - 2\,(u-v)^{2HK}\right]$$

$$\lim_{\varepsilon\to 0}\frac{b_\varepsilon(u,v)}{2^{2HK}} = 0 \quad \text{and} \quad |\frac{b_\varepsilon(u,v)}{2^{2HK}}| \le C$$

Using Taylor expansion and noticing that $a_0(u,v) = 0 \quad \frac{da_\varepsilon(u,v)}{d\varepsilon} = 0$ for every u, v and

$$\frac{da_\varepsilon(u,v)}{d^2\varepsilon^2} = \frac{H^2K(K-1)}{2^{K-3}}\left(u^{2H}+v^{2H}\right)^{K-1}u^{2H-1}v^{2H-1},$$

we obtain for every u, v,

$$a_\varepsilon(u,v) = \frac{H^2K(K-1)}{2^{K-3}}\left(u^{2H}+v^{2H}\right)^{K-1}u^{2H-1}v^{2H-1}\varepsilon^2 + \sigma(\varepsilon^2).$$

This shows that

$$\lim_{\varepsilon\to 0}\frac{a_\varepsilon(u,v)}{\varepsilon} = 0 \quad \text{for every} \quad u,v.$$

□

Remark 3.1. *The above proposition distinguishes a special case which seems to be more interesting than the other cases: the case KH =* $\frac{1}{2}$*. If k = 1, then H =* $\frac{1}{2}$ *and we deal with a wiener process. If K*$\neq 1$*, we have an example of a Gaussian process, having non-trivial quadratic variation which equals* $2^{1-t}t$*, so, modulo a constant, the same as Brownian motion.*

Denote by W a standard Wiener processes.

Proposition 3.3. *The process* $B^{H,K} + W$*, restricted to each compact interval* $[0,T]$*, is equivalent in law with a Wiener process if* $HK > \frac{3}{4}$*.*

Proof. If X is a gaussian process with covariance $R(t,s)$ such that $\frac{\partial^2 R}{\partial_s\partial_t} \in L^2\left([0,T]\right)$, the proces $Y_t = X_t + W_t$ is a semi-martingale equivalent in law to a Wiener process.

Concerning the process $B^{H,K}$, note that for $s \le t$,

$$\frac{\partial^2 R}{\partial_s\partial_t}(s,t) = \frac{1}{2^K}(2HK(K-1)(t^{2H}+s^{2H})^{K-2}(st)^{2H-1} + 2HK(2HK-1)(t-s)^{2HK-2})$$

since

$$\left(t^{2H}+s^{2H}\right)^{K-2} \leq 2^{K-2}\left(st\right)^{H(K-2)}.$$

The first part above belongs to $L^2([0,T])$ for $HK > \frac{1}{2}$, and the second part for $HK > \frac{3}{4}$.

For any $K \in (0,2)$, let $X^K = (X_t^K, t \geq 0)$ be a Gaussian process defined by $X_t^K = \int_0^\infty (1-e^{-rt}) r^{\frac{-1+K}{2}} dW_r$, where $(W_t, t \geq 0)$ is a standard Brownian motion.

This process was introducted in [11] for $K \in (0,1)$ in order to obtain a decomosition of the bifractional Brownian motion with $H \in (0,1)$ and $K \in (0,1)$. □

Proposition 3.4. *Let $B^{H,K}$ a bifractional Brownian motion with parameters $H \in (0,1)$ and $K \in (0,1)$, $B^{H,K}$ be a bifractional Brownian motion with Hurst parameter $HK \in (0,1)$ and $W = (W_t, t \geq 0)$ a standart Brownian motion.*

Let X_t be the process given by (1). If we suppose that $B^{H,K}$ and W are independent's, then processes $\{Y_t = C_1 X^K_{t^{2H}} + B_t^{H,K}, t \geq 0\}$ and $\{C_2 B_t^{H,K}, t \geq 0\}$ have the same distribution, where $C_1 = \sqrt{\dfrac{2^{-K}K}{\Gamma(1-K)}}$ and $C_2 = 2^{\frac{1-K}{2}}$.

Proof. [11] □

Proposition 3.5. *Assume $H \in (0,1)$ and $K \in (1,2)$ with $HK \in (0,1)$. Let $B^{H,K}$ be bifractional Brownian motion and $W = (W_t, t \geq 0)$ a standard Brownian motion. Let $X^{K,H}$ the process defined by $X_t^{HK} = X^K_{t^{2H}} t \geq 0$. If we suppose that B^{HK} and W are independants, then process $B_t^{H,K} = aB_t^{HK} + bX_t^{H,K}$ where $a = \sqrt{2^{1-K}}$ and $b = \sqrt{\dfrac{K(K-1)}{2^K\Gamma(2-K)}}$ is a centered Gaussian process with covariance function*

$$cov(B_t^{H,K}, B_s^{H,K}) = \frac{1}{2^K}\left[(t^{2H}+s^{2H})^K - |t-s|^{2HK}\right].$$

Proof. The process defined by $B_t^{H,K} = aB_t^{HK} + bX_t^{H,K}$ is a centered Gaussian process.

On the other hand, its covariance functions is given by

$$cov(B_t^{H,K}, B_s^{H,K}) = E(B_t^{H,K}B_s^{H,K}) = a^2E(B_t^{HK}B_s^{HK}) + b^2E(X_t^{H,K}X_s^{H,K})$$

$$= a^2(t^{2HK}+s^{2HK}-|t-s|^{2HK}) + \frac{1}{2^K}((t^{2H}+s^{2H})^K - t^{2HK} - s^{2HK})$$

$$= \frac{1}{2}((t^{2H}+s^{2H})^K - |t-s|^{2HK}).$$

Thus the bifractional Brownian motion $B^{H,K}$ with parameter $H \in (0,1)$ and $K \in (1,2)$ such $HK \in (0,1)$ is well defined and it has a decomposition as a sum of fractional Brownian motion and a absolutely continuous process $X^{H,K}$.

Assume that 2HK = 1, Russo and Tudor in [12] proved that if $K \in (0,1)$, the process $B^{H,K}$ is not a semi-martingale.

In the case when $1 < K < 2$, $B^{H,K}$ is a semi-martingale because we have a decomposition of this process as a sum of a brownian motion $B^{\frac{1}{2}}$ and a finite variation process $X^{H,K}$ □

Proposition 3.6. *For $K \in (0,1]$ and $H \in (0,1)$, the process $B^{H,K}$ is not a Markov process.*

Proof. Recall that a Gaussian process with covariance R is Markovian, if, and only if,

$$R(s,u)R(t,t) = R(s,t)R(t,u)$$

for every $s \leq t \leq u$. it is straighforward to check that $B^{H,K}$ does not satisfy this condition. □

Proposition 3.7. *For all constants $0 < a < b$, $B_0^{H,K}$ is strongly locally φ-non deterministic on I = [a, b] with $\varphi(r) = r^{2HK}$. That is, there exist positive constants $C_{2,1}$ and r_0 such that for all $t \in I$ and all $0 < r \leq \min(\{t, r_0\})$,*

$$Var\left(B_0^{H,K}(t)/B_0^{H,K}(s)) : s \in I, r \leq |s-t| \leq r_0\right) \geq C_{2,1}\varphi(r).$$

Proof. We consider the centered stationary Gaussian process $Y_0 = \{Y_0(t), t \in R\}$ defined through the Lamperti's transformation

$$Y_0(t) = e^{-HKt} B_0^{H,K} e^t, \qquad \forall t \in R$$

$r(t) = E(Y_0(0)Y_0(t))$ is given by

$$r(t) = \frac{1}{2^K}((e^{2Ht}+1)^K - |e^t - 1|^{2HK}).$$

Hence $r(t)$ is an even function and by Taylor expansion, we verify that $r(t) = 0(e^{-\beta t})$ as $t \to \infty$ where $\beta = \min\{H(2-k), HK\}$. Thus $r(.) \in L^1(R)$. On the other hand, by using Taylor expansion again, we also have $r(t) \sim 1 - \frac{1}{2^K}|t|^{2HK}$ as $t \to 0$ (3).

By Bochner's theorem, Y_0 has the following stochastic integral representation $Y_0(t) = \int_R e^{i\lambda t} W(d\lambda)\ t \in R$, where w is a complex Gaussian measure Δ whose Fourier transform is r(.)

$$f(\lambda) = \frac{1}{\Pi}\int_0^\infty r(t)cos(\lambda t)dt.$$

So that $f(t) \sim C_{2,2}|\lambda|^{-(1+2HK)}$ as $\lambda \to \infty$; where $C_{2,2} > 0$ is an explicit constant depending only on HK. Hence, by a result of Cuzick, Xiao, $Y_0 = \{Y_0(t), t \in R\}$ is strongly locally φ-nondeterministic on any interval $J = [-T, T]$ with $\varphi(r) = r^{2HK}$ in the sense that exist positive constants δ and $C_{2,3}$ such that exist for all $t \in [-T, T]$ and all $r \in (0, |t| \wedge \delta)$,

$$Var\left(Y_0(t)/Y_0(s)\right) : s \in J, r \leq |s-t| \leq \delta) \geq C_{2,3}\varphi(r).$$

Now we prove the strong local nondeterminism of $B_0^{H,K}$ on I. Note that $B_0^{H,K}(t) = t^{HK} Y_0(lnt)$ for all $t > 0$. We choose $r_0 = a\delta$. Then all s, t $\in I$ with $r \leq |s-t| \leq r_0$. We have

$$\frac{r}{b} \leq |lns - lnt| \leq \delta.$$

For (4), (5) and $r < r_0\ t \in [a, b]$,

$$Var\left(B_0^{H,K}(t)/B_0^{H,K}(s)) : s \in I, r \leq |s-t| \leq r_0\right)$$

$$Var\left(B_0^{H,K}(t)/B_0^{H,K}(s))\right) = Var\left(t^{HK}Y_0(lnt)/s^{HK}Y_0(lns)\right)$$

$$\geq a^{2HK} Var\left(Y_0(lnt)/Y_0(lns) : s \in I, \frac{r}{b} \leq |lns - lnt| \leq \delta\right) \geq C_{2,4}\varphi(r).$$

□

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