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CONSISTENCY OF THE DRIFT PARAMETERS ESTIMATES IN THE FRACTIONAL BROWNIAN DIFFUSION MODEL AND ESTIMATION OF THE HURST PARAMETER BY MAXIMUM LIKELIHOOD METHOD

BA Demba Bocar[1] and Thioune Moussa

ABSTRACT. In this paper, we study the problem of estimating the unknow parameters in a long memory process based on the maximum likelihood method. We consider again a diffusion model involving fractional Brownian motion. Our goal is to study the consistency of the drift parameter estimates depending on the form of the model.

1. INTRODUCTION

The statistical estimation of the Hurst index is one of the fundamental problems in the literature of long-range dependent and self-similar processes the phenomenon of long memory has been noted in nature long before the construction of suitable stochastic models: in fields as diverse as hydrology, economics, chemistry, mathematics, physics, geosciences, and environmental sciences, it is not un common for observations made for apart to be non-trivially correlated.

[1]*corresponding author*

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The mathematical study of long memory processes was initiated by the work of Mandelbrot on self-similar and other stationary stochastic processes that exhibit long-range dependence. He built the fondations for the study of these processes and he was the first one to mathematically define the fractional motion, the the prototype of self-similar and long-range dependent processes.

The problem of the statistical estimation of the self-similarity and long memory parameter H is of great importance.

This parameter determines the mathematical properties of the model and consequently describes the behavior of the underlying physical system.

One can find several techniques related to the Hurst index estimation problem in the literature. There are a lot of graphical methods including the R/S statistic, the correlogram and partial correlations plot, the variance plot and the variogram wich are widely used in geosciences and hydrology.

Several contributions have abready reported for parameter estimation problems concerning continuous times models where the driving processes are fractional Brownian motion(see Kelptsyna and al)

Actually, the problem of maximum likelihood estimation of the drift parameter has also has been extensively studied.

The maximum likelihood technique is chosen in this for two reasons: one is that this technique has been applied efficiently in a large set, the other is that it has well-documented favorable properties, such a being asymptotically consistent un based, efficient and normally distributed about the true parameter values.

2. Preliminary

Let us consider the observations $Y_N = \{B_H(k+1) - B_H(k), k = 0, \ldots, N-1\}$ from a fractional Brownian motion B_H. Denote by $\sum_N$ the covariance matrix of the sample $\{B_H(k+1) - B_H(k), k = 0, \ldots, N-1\}$

$$\begin{aligned}\sum_{k,l} &= E\left([B_H(k+1) - B_H(k)]\,[B_H(l+1) - B_H(l)]\right) \\ &= C\left(|l-k+1|^{2H} - 2\,|l-k|^{2H} + |l-k-1|^{2H}\right)\end{aligned}$$

where $C = \dfrac{\sigma^2}{2} = \dfrac{var(B_1^H)}{2}$. The likelihood of the vector Y_N is given by

$$L_N(H,C) = \frac{1}{(2\pi)^{\frac{N}{2}}\sqrt{det(\sum_N)}} exp\left(-\frac{1}{2} Y_N \sum_N^{-1}{}^t\, Y_N\right)$$

the maximum likelihood estimator is obtained by minimizing $-\ln L_N\left(H, C\right).$

$$\left(\widehat{H}_N, \widehat{C}_N\right) = \underset{H \in (\frac{1}{2},1), C\in(0,+\infty)}{\text{Argmin}} \quad \ln\, det\left(\sum_N\right) + \frac{1}{2} Y_N \sum_N^{-1} t\, Y_N$$

as $N \to +\infty$, the estimator $\left(\widehat{H}_N, \widehat{C}_N\right)$ is strongly consistent and asymptotically Gaussian.(see [9]).

The maximum likelihood method use observations of a fractional Brownian motion path on an un bounded domain $\{B_H(k),\, k = 0, \ldots, N\}$ asymptotically in N.

In this case it is classical to use the quadratic variations on $[0, 1]$ of the process X at scale $\frac{1}{N}$ defined by

$$V_N = \sum_{k=0}^{N-1}\left(X\left(\frac{k+1}{N}\right) - X\left(\frac{k}{N}\right)\right)^2$$

A result in [5] ensures that

$$\lim_{N\to+\infty} N^{2H-1}\, V_N \overset{\text{a.s}}{=} 1$$

the quadratic variations can therefore be used to identify parameter H

$$H_N = \frac{1}{2} + \frac{1}{2}\,\log_2 \frac{V_{\frac{N}{2}}}{V_N}.$$

When considering the limiting distribution of the quadratic variation found two cases

(i) $0 < H < \frac{3}{4}$. The variable $\sqrt{N}\left(N^{2H-1}\, V_N - 1\right)$ converges in distribution, as $N \longmapsto +\infty$ to a Gaussian variable.

(ii) $\frac{3}{4} < H < 1$. The variable $N^{2-2H}\left(N^{2H-1}\, V_N - 1\right)$ converges in distribution as $N \longmapsto +\infty$ to a non-Gaussian variable.

Therefore, the rate of convergence of the estimator H based on quadratic variations dramatically fails when $\frac{3}{4} < H < 1$.

3. Some results

3.1. **Hypothesis.** Let $g(t,\xi)$ be the following harmonizable fractional integred type Kernel $g(t,\xi)=\dfrac{a(t)}{|\xi|^{\frac{1}{2}+H}}+\varepsilon(t,\xi)$ with $a(t)\in C^2 \quad \varepsilon(t,\xi)\in C^{2,2}$ satisfying for i, j = 0, 1, 2.

$$\left|\frac{\partial^{i+j}}{\partial t^i\,\partial\xi^j}\varepsilon\,(t,\xi)\right|\leqslant\frac{C}{|\xi|^{\frac{1}{2}+\eta+j}}\quad with\quad \eta>H\quad and\quad \overline{\varepsilon(t,\xi)}=\varepsilon(t,-\xi).$$

Here C denotes a generic constant that can change from an occurrence to another.

Theorem 3.1. *Let X be a process satisfying Hypothesis*

1. *Strong consistency* $\lim\limits_{N\to+\infty}\widehat{H}_N\overset{a.s}{=}H.$
2. *Asymptotic normality if in Hypothesis* $\eta>\frac{1}{2}+H$ *as* $N\longmapsto+\infty$*,* $\sqrt{N}\left(\widehat{H_N}-H\right)$ *converges in distribution to a centered Gaussian variable.*

To prove the theorem, we use the following lemma.

Lemma 3.1. *Let us*

$$I(S,S^{'})=\int_{\mathbb{R}}\sum_{k,k^{'}=0}^{K}a_k\,a_{k'}\,S\left(\frac{k+p}{N},N_u\right)\,S^{'}\left(\frac{k^{'}+p^{'}}{N},N_u\right)$$
$$\times\,e\left((k+p)\,u\right)\,\overline{e}\left(\left(k^{'}+p^{'}\right)u\right)N\,du.$$

So that

$$S\,(t,\xi)=\frac{a(t)}{|\xi|^{H+\frac{1}{2}}}\quad S^{'}\,(t,\xi)=\frac{a^{'}(t)}{|\xi^{'}|^{H+\frac{1}{2}}}\quad with\quad a^{'}(t)\in C^2,$$

the following bound $|I(S,S^{'})_{p,p'}|\leqslant\dfrac{C}{N^{\delta+\delta'}\left(1+(p-p')^2\right)}$ *holds for N large enough if* $\left|\dfrac{\partial^{i+j}}{\partial t^i\,\partial\xi^j}\,S\,(t,\xi)\right|\leqslant\dfrac{C}{|\xi|^{\frac{1}{2}+\delta+j}}.$

Proof. We use a Taylor expansion of S of order 2 for $0\leq k\leq K$:

$$S\left(\frac{k+p}{N},N_u\right)=\sum_{j=0}^{2}\frac{\partial^j}{\partial^j t}S(\frac{P}{N},Nu)\frac{k^j}{N^j j!}+\frac{k^2}{2N^2}\frac{\partial^2}{\partial t^2}S\left(\frac{k+p}{N},N_u\right),$$

where $0 \leq k \leq K$. the same holds for S^t. We then obtain the following expansion for $I(S,S^{'})_{p,p'}$:

$$
\begin{aligned}
I(S,S^{'})_{p,p'} &= \sum_{j=0}^{2} N^{-1} \sum_{j_1+j_2=j} \frac{1}{j_1! j_2!} \\
&\times \int_R \left[\sum_{K,K'}^{K} a_k a_{k'} k^{j_1} k^{'j_2} e((k+p)u) \bar{e}((k^{'}+p^{'})u) \right] \\
&\times \frac{\partial^{j_1}}{\partial t^{j_1}} S(\frac{P}{N}, Nu) \frac{\partial^{j_2}}{\partial t^{j_2}} S(\frac{P^{'}}{N}, Nu) N du \\
&\times \sum_{j=2}^{4} N^{-j} \sum_{j_1+j_2=j} \frac{1}{j_1! j_2!}
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
&\times \int_R \sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k^{'j_2} e((k+p)u) \bar{e}((k^{'}+p^{'})u) \\
&\times \frac{\partial^{j_1}}{\partial t^{j_1}} S(\frac{\epsilon(j_1)k+p}{N}, Nu) \frac{\partial^{j_2}}{\partial t^{j_2}} S^{'}(\frac{\epsilon(j_2)k^{'}+p^{'}}{N}, Nu) N du.
\end{aligned} \tag{3.2}
$$

where $\epsilon(j) = 1$ if j=2, 0 otherwise.

Let us first consider the case $p = p^{'}$. We have to bound $|I(S,S^{'})_{p,p'}|$ by $CN^{-\delta-\delta^{'}}$.

We clearly have

$$
\sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k^{'j_2} e((k+p)u) \bar{e}((k^{'}+p)u) = \sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k^{'j_2} e^{iu((k-k^{'})}
$$

Each integral of (3.1) is bounded by a term involving

$$
\int_R |\sum_{K,K'=0}^{K} a_k a_{k'} k^{j_1} k^{'j_2} e^{iu((k-k^{'})}| \frac{du}{|u|^{\delta+\delta^{'}+1}}. \tag{3.3}
$$

The function $\sum_{K,K'=0}^{K} a_k a_{k'} k^{j_1} k^{'j_2} e^{iu((k-k^{'})}$ and its derivates up to order 2 vanish at u = 0 hence $\sum_{K,K'=0}^{K} a_k a_{k'} k^{j_1} k^{'j_2} e^{iu((k-k^{'})} = 0(u^2)$ when $|u| \to 0$. Then

$$
\int_R |\sum_{K,K'=0}^{K} a_k a_{k'} k^{j_1} k^{'j_2} e^{iu((k-k^{'})}| \frac{du}{|u|^{\delta+\delta^{'}+1}} < +\infty,
$$

since $\delta+\delta^{'}<2$.Moreover, since $\delta+\delta^{'}>0$ the integral (1.3) is convergent at infinity and therefore each term of line (3.1) is an $0(N^{-j-\delta-\delta^{'}})$. It remains to bound (3.2). Each integral of (3.2) is bounded by

$$CN^{-\delta-\delta^{'}}|\sum_{K,K^{'}=0}^{K}a_{k}a_{k^{'}}k^{j_1}k^{'j_2}|\int_{R}|e((k+p)u)e((k^{'}+p)u)|\frac{du}{|u|^{\delta+\delta^{'}+1}}.$$

As $|u|\to\infty, e((k+p)u)=0(1)$ hence

$$\int_{1}^{+\infty}|e((k+p)u)e((k^{'}+p)u)|\frac{du}{|u|^{\delta+\delta^{'}+1}}=0(1)$$

and as $|u|\to 0, e((k+p)u)=0((k+p)u)$ hence

$$\int_{0^+}^{1}|e((k+p)u)e((k^{'}+p^{'})u)|\frac{du}{|u|^{\delta+\delta^{'}+1}}=0(pp^{'}).$$

Since $p<N$, (3.2) is bounded by $0(N^{-\delta-\delta^{'}})$. Lemma 3.1 is proved for $p=p^{'}$. It remains to prove Lemma 1 when $p\neq p^{'}$. Expression (3.1) leads to the integral factor

$$\int_{R}e^{iu((p-p^{'})}\left[\sum_{K,K^{'}=0}^{K}a_{k}a_{k^{'}}k^{j_1}k^{'j_2}e^{iu((k-k^{'})}\right]\frac{\partial^{j_1}}{\partial t^{j_1}}S(\frac{P}{N},Nu)\frac{\partial^{j_2}}{\partial t^{j_2}}S^{'}(\frac{p^{'}}{N},Nu)Ndu.$$

We integrate by parts twice and this gives

$$\int_{R}\frac{e^{iu((p-p^{'})}}{(p-p^{'})^2}\frac{\partial^2}{\partial u^2}\left[\left(\sum_{k,k^{'}=0}^{K}a_{k}a_{k^{'}}k^{j_1}k^{'j_2}e^{iu(k-k^{'})}\right)\frac{\partial^{j_1}}{\partial t^{j_1}}S(\frac{P}{N},Nu)\frac{\partial^{j_2}}{\partial t^{j_2}}S^{'}(\frac{p^{'}}{N},Nu)\right]Ndu.$$

To prove that the previous integral converges, and that all terms coming from integrated terms in the integration by parts vanish, we only have to prove the absolute convergence of the terms given by the second derivative with respect to u of

(3.4) $$\psi(u,j_1,j_2)\frac{\partial^{j_1}}{\partial t^{j_1}}S(\frac{P}{N},Nu)\frac{\partial^{j_2}}{\partial t^{j_2}}S^{'}(\frac{p^{'}}{N},Nu),$$

where $\psi(u,j_1,j_2)=\sum_{K,K^{'}=0}^{K}a_{k}a_{k^{'}}k^{j_1}k^{'j_2}e^{iu((k-k^{'})}$. Clearly, as $|u|$ goes to ∞. $|\frac{\partial^i}{\partial u^i}\psi(u,j_1,j_2)|=0(1)$ for i = 0, 1, 2. This implies the convergence of (3.4) as $|u|\to\infty$. to have convergence when $|u|\to 0$ let us remark that $|\frac{\partial^i}{\partial u^i}\psi(u,j_1,j_2)|=$

$0(|u|^{4-i})$, when $|u| \to 0$. Then

$$|\frac{\partial^{i_1}\psi(u, j_1, j_2)}{\partial u^{i_1}} \frac{\partial^{i_1+j_2}}{\partial t^{j_1}\partial u^{i_2}} S(\frac{P}{N}, Nu) \frac{\partial^{i_3+j_2}}{\partial t^{j_2}\partial u^{i_3}} S^{'}(\frac{p^{'}}{N}, Nu)|$$

$$\leq \frac{C|u|^{4-(i_1+i_2+i_3)-(\delta+\delta^{'}+1)}}{N^{\delta+\delta'+1+i_2+i_3}}.$$

For $i_1 + i_2 + i_3 = 2$. Hence each term of the first line (cf. 3.1) of the expansion of $I(S, S^{'})_{p,p'}$ is of order $\dfrac{1}{N^{\delta+\delta'+j}(p-p')^2}$. We use a similar upper bound $0(N^{\delta+\delta^{'}-2})$ for the second line (cf.(3.2)) of the expansion of $I(S, S^{'})_{p,p'}$. Since $p, p^{'} < N$,we have proved Lemma 1 for $p \neq p^{'}$.

A second technical lemma relates the asymptotic behavior of $I(S, S^{'})_{p,p'}$ when $S(t, \xi) = \frac{a(t)}{|\xi|^{H+\frac{1}{2}}}$ to the function

$$F_\gamma(x) = \int_R \sum_{k,k'=0}^{K} a_k a_{k'} \frac{e^{i(x+k-k^{'})u}}{|u|^{\gamma+1}} du.$$

□

Lemma 3.2. *if a is* C^2 *and* $S(t, \xi) = \dfrac{a(t)}{|\xi|^{H+\frac{1}{2}}}$,

$$I(S, S^{'}) = N^{-\delta-\delta^{'}}\, a\,(\Delta p)\; a(\Delta p^{'})\, F_{\delta+\delta'}\left(p-p^{'}\right) + 0\, \left(\frac{1}{N^{\delta+\delta'+1}\left(1+(p-p')^{2}\right)}\right).$$

Proof. To begin with, we use the Taylor expansion of a at order 2 to get the expansion (3.1) and (3.2)

$$I_{p,p'} = \int_R \left[\sum_{k,k'=0}^{K} a_k a_{k'} e((k+p)u)\bar{e}((k^{'}+p^{'})u)\right] \frac{a\,(\Delta p)\; a\left(\Delta p^{'}\right)}{N^{\delta+\delta'}|u|^{\delta+\delta'+1}} du$$
$$+ 0\, \left(\frac{1}{N^{\delta+\delta'+1}\left(1+(p-p')^{2}\right)}\right).$$

Using

$$\sum_{k,k'=0}^{K} a_k a_{k'} e((k+p)u)\bar{e}((k^{'}+p^{'})u) = \left(\sum_{k,k'=0}^{K} a_k a_{k'} e^{i(k-k^{'})u}\right) e^{i(p-p^{'})u}.$$

and lemma follows. □

Note that

$$\begin{aligned} EV_N &= \sum_{p=0}^{N-K} E\left(\sum_{k=0}^{K} a_k\, X\left(\frac{k+p}{N}\right)\right)^2 \\ &= \sum_{p=0}^{N-K} \int_{\mathbb{R}} \left|\sum_{k=0}^{K} a_k\, g\left(\frac{k+p}{N}, \xi\right)\, e\left(\xi \frac{k+p}{N}\right)\right|^2 d\xi, \end{aligned}$$

where function e stands for $e(\xi) = e^{i\xi} - 1$.

Using the expansion for $g(t,\xi)$, we can express EV_N as a sum of terms, each of them being of the following form $(p = 0, \ldots, N-K)$,

$$I(S, S^{'})_{p,p'} = \int_{\mathbb{R}} \sum_{k,k^{'}=0}^{K} a_k\, a_{k'}\, S\left(\frac{k+p}{N}, N_u\right)\, S^{'}\left(\frac{k^{'}+p^{'}}{N}, N_u\right)$$

$S\left(t,\xi\right) \in C^{2,2}\left([0,1] \times \mathbb{R}^*\right)$ and $\left|\frac{\partial^{i+j}}{\partial t^i \partial \xi^j}\, S\left(t,\xi\right)\right| \leqslant \frac{C}{|\xi|^{\frac{1}{2}+\delta+j}}$ for i=0 to 2 and j=0 to 2 with $0 < \delta < 1$, and the same hold for $S^{'}$.

Since X is a Gaussian process, the variance V_N is given by

$$\begin{aligned} Var\left(V_N\right) = 2 \sum_{p,p^{'}=0}^{N-K} & \left(\int_{\mathbb{R}} \sum_{k,k^{'}=0}^{K} a_k\, a_{k'}\, g\left(\frac{k+p}{N}, \xi\right)\, g\left(\frac{k^{'}+p^{'}}{N}, \xi\right)\right. \\ & \left. \times e\left(\xi \frac{k+p}{N}\right)\, \overline{e}\left(\xi \frac{k^{'}+p^{'}}{N}\right) d\xi\right)^2 \end{aligned}$$

which is a sum of term $I(S, S^{'})_{p,p'}$.

The estimation of the expectation and variance deduce from the Lemma 3.1 and Lemma 3.2.

Consistency of the drift parameter estimates in the pure fractional Brownian Diffusion Model.

First we consider the ”pure” fractional diffusion model and establish strong consistency and asymptotic normality of the maximum likelihood drift parameter estimate.

We assume that the fBm B_t^H with $H \in (\frac{1}{2}, 1)$ is define on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $(\mathcal{F}_t)_{t \geqslant 0}$ the filtration generated by B_t^H.

Considerer a diffusion equation containing a stochastic differential driven by B^H.

$$dX_t = \theta\, a\,(t, X_t)\, dt + b\,(t, X_t)\, dB_t^H \qquad X_{t=0} = X_0 \,\in\, \mathbb{R}, \tag{3.5}$$

with $a(t, X_t)$ and $b(t, X_t)$ be two stochastic processes.

Differential equation can be rewritten in the integral form

$$X_t = X_0 + \theta \int_0^t a\,(s, X_s)\; ds + \int_0^t b\,(s, X_s)\; dB_t^H \qquad t \,\in\, [0, T]\,, \tag{3.6}$$

where X_0 is a random variable. The stochastic integral $\int_0^t a\,(s, X_s)\; ds$ is an ordinary Riemann-Stieltjes integral for each X_t while $\int_0^t b\,(s, X_s)\; dB_s^H$ is defined as that given by Dai and Heyde [8].

Generally speaking, the integral $\int_0^t a\,(s, X_s)\; ds$ exists under standard conditions on $a(s, X_t)$. The integral $\int_0^t b\,(s, X_s)\; dB_s^H$ exists only under the conditions given in Dai and Heyde [8] for defining stochastic integrals with respect to $B_H(t)$.

Suppose that the equation has unique pathwise solution. Now, let $T > 0$ be fixed. We are in a position to find the likelihood ratio $\dfrac{dP_\theta(t)}{dP_0(t)}$ for the probability measure $P_\theta(t)$ corresponding to our model and the probability measure $P_0(t)$ corresponding to the model with zero drift. Suppose that the following assumption holds:

(i) $b\,(t, X_t) \neq 0, \quad t \,\in\, [0, T]$ and $\dfrac{a\,(t, X_t)}{b\,(t, X_t)}$ is a s lebesgue integrable on [0, T]. Denote $\varphi_t = \dfrac{a\,(t, X_t)}{b\,(t, X_t)}$ and introduce the new process $\widehat{B}_t^H = B_t^H + \theta \displaystyle\int_0^t \varphi_s\, ds$;

(ii) $\theta \displaystyle\int_0^t l_H(t, s) |\varphi(s)|\, ds < \infty, \quad t \in [0, T]$
$l_H(t, s)$ is the likelihood ratio;

(iii) $\theta \displaystyle\int_0^t l_H(t, s)\, \varphi(s)\, ds \;=\; \tilde{\alpha} \int_0^t \delta_s ds, \quad t \,\in\, [0, T]$ with $\tilde{\alpha} = (1 - 2\alpha)^{\frac{1}{2}}$, $\alpha = H - \frac{1}{2}$;

(iv) $E \displaystyle\int_0^t s^{2\alpha}\, \delta_s^2\, ds < \infty \quad t \,\in\, [0, T]$;

(v) $E\,exp\left\{L_t - \frac{1}{2}\left\langle L\right\rangle_t\right\} = 1$ the process $\widehat{B}_t^H$ is an fBm on [0, T] w.r.t the measure Q defined via the relation

$$\frac{dP_\theta(t)}{dP_0(t)} = \exp\left\{L_t - \frac{1}{2}\left\langle L\right\rangle_t\right\}, \quad t\in[0,T] \quad with \quad L_t = \int_0^t s^\alpha\,\delta_s\,d\widehat{B}_s.$$

In order to find the maximum likelihood estimate of the parameter θ, we use likelihood ratio which can be write as

$$\frac{dP_\theta(t)}{dP_0(t)} = \exp\left\{\int_0^t s^\alpha\,\delta_\alpha d\widehat{B}_s - \frac{1}{2}\int_0^t s^{2\alpha}\,\delta_s^2\,ds\right\}$$

where δ_s is defined according to the integral representation iii).

Denote $\Psi(t,x) = \dfrac{a\,(t,x)}{b\,(t,x)}$ so that $\Psi(t,X_t) = \varphi(t) \quad I(t) = \int_0^t l_H(t,s)\varphi(s)\,ds.$

Theorem 3.2. *Let* $\Psi(t,s)\in C^1[0,T]\bigcap C^2(\mathbb{R})$. *Then for* $t>0$

$$\begin{aligned}I'(t) &= C(H)\,\Psi(0,0)\,t^{-2\alpha} + \int_0^t l_H(t,s)\left(\Psi_s'(s,X_s) + \theta\,\Psi_x'(s,X_s)\,a(s,X_s)\right)ds\\ &- \alpha\,C_H^{(5)}\int_0^t s^{-1-\alpha}(t-s)^{-\alpha}\int_0^s\left(\Psi_t'(u,X_u) + \theta\,\Psi_x'(u,X_u)\,a(u,X_u)\right)du\,ds\\ &+ (1-2\alpha)\,C_H^{(5)}\,t^{-2\alpha}\int_0^t s^{2\alpha-2}\int_0^s u^{1-\alpha}\,(s-u)^{-\alpha}\,\Psi_x'(u,X_u)\,b(u,X_u)dB_u^H\,ds\\ &+ C_H^{(5)}\,t^{-1}\int_0^t u^{1-\alpha}(t-u)^{-\alpha}\,\Psi_x'(u,X_u)\,b(u,X_u)\,dB_u^H\end{aligned}$$

where $C(H) = (1-2\alpha)\,B(1-\alpha,1-\alpha)\,C_H^{(5)}$.

Proof. According to the Ito formula

$$\begin{aligned}\varphi_s &= \Phi(0,0) + \int_0^s\left(\Psi_t'(u,X_u) + \Psi_x'(u,X_u)\,\theta\,a(u,X_u)\right)\,du\\ &+ \int_0^s \Psi_x'(u,X_u)\,b(u,X_u)\,dB_u^H\\ X_t - X_{t_0} &= \int_{t_0}^t \theta\,a(u,X_u)du + \int_{t_0}^t b(u,X_u)dB_H(u),\end{aligned}$$

where the first integral is an ordinary Riemann-Stieltjes integral for each $\omega\in\Omega$, while the second is an Ito integral defined in Dai et Heyde [8].

Assume that a two variable function $\Psi(u, X_u) : [0,T] \times \mathbb{R} \to \mathbb{R}$ has uniformly continuous partial derivatives $\frac{\partial \Psi}{\partial t}$, $\frac{\partial \Psi}{\partial x}$ and $\frac{\partial^2 \Psi}{\partial x^2}$. Assume further that

$$\sup_{0 \le u \le T} \mathbb{E} \left| \Psi(u, X_u) \right|^2 < \infty,$$

$$\sup_{0 \le u \le T} \mathbb{E} \left| \frac{\partial \Psi}{\partial t}(u, X_u) \right|^2 < \infty,$$

$$\sup_{0 \le u \le T} \mathbb{E} \left| \frac{\partial \Psi}{\partial x}(u, X_u) \right|^2 < \infty,$$

$$\sup_{0 \le u \le T} \mathbb{E} \left| \frac{\partial^2 \Psi}{\partial x^2}(t, X_t + O_{L_2}(1)) \right|^2 < \infty,$$

$$\sup_{0 \le u \le T} \mathbb{E} \left| a(t) \right|^2 < \infty,$$

$$\sup_{0 \le u \le T} \mathbb{E} \left| b(t) \right|^2 < \infty,$$

$$\mathbb{E} \left| b(t) - b(s) \right| < const \left| t - s \right|^{\beta}, \beta \ge 0,$$

where $O_{L_2}(1)$ means a term such that $\mathbb{E} \left| O_{L_2}(1) \right|^2 < \infty$. Let $\Psi_u = \Psi(u, X_u)$. If, for all $0 \le u \le T$,

$$\int_0^t b(u, X_u) \frac{\partial \Psi}{\partial x}(u, X_u) dB_H(u),$$

exists in the sense described in Dai and Heyde [8], then the following holds

$$\begin{aligned} \varphi_s - \varphi_0 = & \int_0^t \left\{ \frac{\partial \Psi}{\partial x}(u, X_u) + \theta\, a(u, X_u) \frac{\partial \Psi}{\partial x}(u, X_u) \right\} du \\ & + \int_{t_0}^t b(u, X_u) \frac{\partial \Psi}{\partial x}(u, X_u) dB_H(u), \end{aligned}$$

or, equivalently,

$$d\varphi_s = \left\{ \frac{\partial \Psi}{\partial x}(u, X_u) + \theta\, a(u, X_u) \frac{\partial \Psi}{\partial x}(u, X_u) \right\} du + b(u, X_u) \frac{\partial \Psi}{\partial x}(u, X_u) dB_H(u).$$

Substituting the above into the integral $I(t) = \int_0^t l_H(t,s)\varphi_s\,ds$ we obtain

$$\begin{aligned} I(t) &= c(H,1)\Psi(0,0)t^{1-2\alpha} + \int_0^t l_H(t,s)\int_0^s \Psi_t^{'}(u,X_u)\,du\,ds \\ &+ \theta\int_0^t l_H(t,s)\int_0^s \Psi_x^{'}(u,X_u)\,a(u,X_u)duds \\ &+ \int_0^t l_H(t,s)\int_0^s \Psi_x^{'}(u,X_u)\,b(u,X_u)dB_u^H\,ds, \end{aligned}$$

$C(H,1) = C_H^{(5)}\,B(1-\alpha,1-\alpha)$ and now our aim is to differentiate I(t),

$$\begin{aligned} s^{-\alpha}\int_0^s \Psi_x^{'}(u,X_u)\,a(u,X_u)du &= \int_0^s u^{-\alpha}\Psi_x^{'}(u,X_u)\,a(u,X_u)du \\ - \alpha\int_0^t u^{-1-\alpha}\int_0^u \Psi_x^{'}(u,X_u)\,a(v,X_v)dvdu. \end{aligned} \tag{3.7}$$

According to representation given above, there exist a.s the fractional derivatives of order α, i.e the derivatives of fractional integrals:

$$\begin{aligned} &\frac{d}{dt}\int_0^t l_H(t,s)\int_0^s \Psi_x^{'}(u,X_u)duds = \int_0^t l_H(t,s)\Psi_t^{'}(s,X_s)ds \\ &- \alpha\,C_H^{(5)}\int_0^t s^{-1-\alpha}(t-s)^{-\alpha}\int_0^s \Psi_t^{'}(u,X_u)duds \\ &\frac{d}{dt}\int_0^t l_H(t,s)\int_0^s \Psi_x^{'}(u,X_u)\,a(u,X_u)duds \\ &= \int_0^t l_H(t,s)\Psi_x^{'}(s,X_s)\,a(s,X_s)ds \\ &- \alpha\,C_H^{(5)}\int_0^t s^{-1-\alpha}(t-s)^{-\alpha}\int_0^t \Psi_x^{'}(u,X_u)\,a(u,X_u)duds \end{aligned}$$

Further, it follows that

$$\int_0^t l_H(t,s)\int_0^s \Psi_x^{'}(u,X_u)\,b(u,X_u)dB_u^H\,ds$$

$$= C_H^{(5)}t^{1-2\alpha}\int_0^t s^{2\alpha-2}\int_0^s u^{1-\alpha}(s-u)^{-\alpha}\Psi_x^{'}(u,X_u)\,b(u,X_u)dB_u^H\,ds,$$

the proof follows immediately from the previous relations.

Now, we can write

$$\frac{dP_\theta(t)}{dP_0(t)} = exp\left\{\frac{\theta}{\tilde{\alpha}}\int_0^T s^\alpha I^{'}(\alpha)d\widehat{B}_s - \frac{\theta^2}{2(1-2\alpha)}\int_0^T s^{2\alpha}(I^{'}(s))^2 ds\right\}. \tag{3.8}$$

It follows from the above that the maximum likelihood estimate is achieved under the condition

$$\int_0^T s^\alpha I^{'}(s)dB^s - \frac{\theta}{\tilde{\alpha}}\int_0^T s^{2\alpha}(I^{'}(s))^2 ds = 0$$

whence

$$\tilde{\theta}_t = \frac{\tilde{\alpha}\displaystyle\int_0^t s^\alpha I^{'}(s)d\hat{B}^s}{\displaystyle\int_0^t s^{2\alpha}(I^{'}(s))^2 ds}.$$

Using Theorem 3.2, we obtain

$$\tilde{\theta}_t = \theta + \frac{\tilde{\alpha}\int_0^t s^\alpha I^{'}(s)dB_s}{\int_0^t s^{2\alpha}(I^{'}(s))^2 ds}. \tag{3.9}$$

□

Theorem 3.3. *Let i)— v) hold for any $T > 0$, and morever vi)* $\displaystyle\int_0^t s^{2\alpha}(I^{'}(s))^2 ds = \infty \quad a.s$ *then the maximum likelihood estimate $\hat{\theta}_T$ is strongly consistent as $T \to \infty$*

Proof. From representation (9) and the fact that $\dfrac{X_t}{\langle X\rangle_t} \to 0$ a.s, if X_t is a square integrable martingale and $\langle X\rangle_\infty \to \infty$ a.s. In other words,

$$\frac{\int_0^t s^\alpha I^{'}(s)dB_s}{\int_0^t s^{2\alpha}(I^{'}(s))^2 ds} \to 0,\ t \to \infty\ with\ P_\theta - probability\ 1.$$

□

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DEPARTMENT OF MATHEMATICS
UNIVERSITY IBA DER THIAM OF THIES
THIES,
SENEGAL.
Email address: dbba@univ-thies.sn

DEPARTMENT OF MATHEMATICS
UNIVERSITY IBA DER THIAM OF THIES
THIES,
SENEGAL.
Email address: thiounemsa@gmail.com