

STUDY OF MODELS WITHOUT JUMPS WITH RESPECT TO FRACTIONAL BROWNIAN MOTION

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ABSTRACT. In this paper, we study some models without jumps of stochastic differential equations directed by a fractional Brownian motion.

1. INTRODUCTION

Stochastic differential equations model stochastic evolution as time evolves. These models have a variety of applications in many disciplines and emerge naturally in the study of many phenomena. Examples of these applications are physics, astronomy, mechanics, economics, mathematical finance, geology, genetic analysis, ecology, cognitive psychology, neurology, biology, biomedical sciences, epidemiology, political analysis and social processes, and many other fields of science and engineering.

In general, the study of the stochastic differential equations (SDE) heavily depends on the definition of stochastic integrals. In this work we use Fractional Brownian motion (FBM) with Hurst index $\frac{1}{2} < H < 1$ which is not a semimartingale. Consequently, the standard Itô calculus is not available for stochastic integrals with respect to FBM as an integrator if $\frac{1}{2} < H < 1$.

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The $It\hat{o}$ stochastic calculus has become a fundamental part of modern probability theory and found substantial application in other disciplines. For example, in mathematical finance, $It\hat{o}$'s calculus is a powerful tool for dealing with stock price behavior. Stochastic differential equations driven by Fractional Brownian motion are routinely used to model the dynamics of stock market prices. Fractional Brownian motion (FBM) provides a suitable generalization of Brownian motion. The feature, which most distinguishes FBM from Brownian motion, is that FBM is no longer a semimartingale for $\frac{1}{2} < H < 1$.

We are concerned with definition of some models of stochastic differential equations with respect to fractional Brownian motion (FBM) and to prove of the existence and uniqueness of their solutions.

2. PRELIMINARY

When we consider stochastic differential equations driven by Brownian motion

$$(2.1) \quad dX = b(t, X_t)dt + \sigma(t, X_t)dB(t),$$

$It\hat{o}$'s formula is a powerful tool for dealing with their calculus. When we are concerned with stochastic differential equations driven by fractional Brownian motion

$$(2.2) \quad dX = b(t, X_t)dt + \sigma(t, X_t)dB^H(t),$$

we have noticed that a version of $It\hat{o}$'s formula plays the same role in dealing with equation (2.2). The aim of this section is the following theorem.

Theorem 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $B_h(\tau)$ be a fractional brownian motion on $[0, T]$ such that $\frac{1}{2} < H < 1$ and $B_H(0) = 0$ a.e. (therefore $\mathbb{E}B_H(\tau) \equiv 0$ for any $t \in [0, T]$). Assume stochastic processes $b(\tau, \omega), \sigma(\tau, \omega)$ and $X(\tau, \omega)$ are such that for any $[t_0, t] \subseteq [0, T]$,*

- (1) $b(\tau, \omega)$ is Riemann-Stieltjes integrable on $[0, T]$ for each $\omega \in \Omega$;
- (2) $\int_0^t \sigma(\tau)dB_H(\tau)$ exists in the sense described in Dai and Heyde [3];
- (3) Either of the following holds:

- (a) For any $0 \leq s \leq t_1 \leq t_2, t_3 \leq t_4 \leq T, \{\sigma(\tau) : 0 \leq \tau \leq T\}$ and $\{B_H(\tau) : 0 \leq \tau \leq T\}$ are such that

$$\begin{aligned}
 (2.3) \quad & \mathbb{E} \left\{ (\sigma(t_1) - \sigma(s)) (\sigma(t_3) - \sigma(s)) \right. \\
 & \left. (B_H(t_2) - B_H(t_1)) (B_H(t_4) - B_H(t_3)) \right\} \\
 & = \mathbb{E} \{ (\sigma(t_1) - b(s)) (\sigma(t_3) - \sigma(s)) \} \\
 & \mathbb{E} \{ (B_H(t_2) - B_H(t_1)) (B_H(t_4) - B_H(t_3)) \},
 \end{aligned}$$

or,

- (b) the second derivative $\frac{d^2 b(t)}{dt^2}$ exists, and for any $0 \leq t_1 \leq t_2, t_3 \leq t_4 \leq T$,

$$\{\sigma'(\tau) = \frac{d\sigma(t)}{dt} : s \leq \tau \leq \max\{t_1, t_3\}\}$$

and

$$(B_H(t_1), B_H(t_2), B_H(t_3), B_H(t_4)),$$

such that for any random variables ξ and η such that ξ and η are measurable with respect to $\sigma\{b'(\tau) : s \leq \tau \leq \max\{t_1, t_3\}\}$ et $\mathbb{E} |\xi|^4 < \infty$, the following holds

$$\begin{aligned}
 (2.4) \quad & \mathbb{E} \left\{ \left(\sigma'(s)(t_1 - s) + \xi \right) \left(\sigma'(s)(t_3 - s) + \eta \right) \right. \\
 & \left. (B_H(t_2) - B_H(t_1)) (B_H(t_4) - B_H(t_3)) \right\} \\
 & = \mathbb{E} \{ \left(\sigma'(s)(t_1 - s) + \xi \right) \left(\sigma'(s)(t_3 - s) + \eta \right) \} \\
 & \times \mathbb{E} \{ (B_H(t_2) - B_H(t_1)) (B_H(t_4) - B_H(t_3)) \},
 \end{aligned}$$

and furthermore,

$$(2.5) \quad \sup_{0 \leq t \leq T} \mathbb{E} \left| \frac{d\sigma}{dt}(t, \omega) \right|^4 < \infty, \sup_{0 \leq t \leq T} \mathbb{E} \left| \frac{d^2 \sigma}{dt^2}(t, \omega) \right|^4 < \infty;$$

(4)

$$(2.6) \quad X_t - X_{t_0} = \int_{t_0}^t b(\tau, \omega) d\tau + \int_{t_0}^t \sigma(\tau, \omega) dB_H(\tau),$$

where the first integral in (2.6) is an ordinary Riemann-Stieljes integral for each $\omega \in \Omega$, while the second is an Itô integral defined in Dai et Heyde [3].

Assume that a two variable function $U(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ has uniformly continuous partial derivatives $\frac{\partial U}{\partial t}$, $\frac{\partial U}{\partial x}$ and $\frac{\partial^2 U}{\partial x^2}$. Assume further that

$$(2.7) \quad \sup_{0 \leq t \leq T} \mathbb{E} |U(t, X_t)|^2 < \infty,$$

$$(2.8) \quad \sup_{0 \leq t \leq T} \mathbb{E} \left| \frac{\partial U}{\partial t}(t, X_t) \right|^2 < \infty,$$

$$(2.9) \quad \sup_{0 \leq t \leq T} \mathbb{E} \left| \frac{\partial U}{\partial x}(t, X_t) \right|^2 < \infty,$$

$$(2.10) \quad \sup_{0 \leq t \leq T} \mathbb{E} \left| \frac{\partial^2 U}{\partial x^2}(t, X_t + O_{L_2}(1)) \right|^2 < \infty,$$

$$(2.11) \quad \sup_{0 \leq t \leq T} \mathbb{E} |b(t)|^2 < \infty,$$

$$(2.12) \quad \sup_{0 \leq t \leq T} \mathbb{E} |\sigma(t)|^2 < \infty,$$

$$(2.13) \quad \mathbb{E} |\sigma(t) - \sigma(s)| < \text{const} |t - s|^\beta, \beta \geq 0,$$

where $O_{L_2}(1)$ means a term such that $\mathbb{E} |O_{L_2}(1)|^2 < \infty$. Let $U_t = U(t, X_t)$. If, for any $0 \leq t \leq T$,

$$\int_0^t \sigma(\tau, \omega) \frac{\partial U}{\partial x}(\tau, X_\tau) dB_H(\tau)$$

exists in the sense described in Dai and Heyde [3], then the following holds

$$(2.14) \quad \begin{aligned} Y_t - Y_{t_0} &= \int_0^t \left\{ \frac{\partial U}{\partial x}(\tau, X_\tau) + b(\tau, \omega) \frac{\partial U}{\partial x}(\tau, X_\tau) \right\} d\tau \\ &+ \int_{t_0}^t \sigma(\tau, \omega) \frac{\partial U}{\partial x}(\tau, X_\tau) dB_H(\tau) \end{aligned}$$

or equivalently,

$$(2.15) \quad dY_t = \left\{ \frac{\partial U}{\partial x}(t, X_t) + b(t, \omega) \frac{\partial U}{\partial x}(t, X_t) \right\} dt + \sigma(t, \omega) \frac{\partial U}{\partial x}(t, X_t) dB_H(t).$$

Remark 2.1.

(1) Since $\mathbb{E} (B_H(t + \Delta) - B_H(t))^2 = |\Delta|^{2H}$, where $2H > 1$, there is no term

$$\frac{1}{2} b^2(\tau, \omega) \frac{\partial^2 U}{\partial x^2}(\tau, X_t) d\tau$$

in (2.14), in contrast to that of the usual Itô formula with respect to Brownian motion.

(2) The requirements on $(\tau), b(\tau), X(\tau)$ and $U(\tau, X_\tau)$, such as conditions 1, 2 and 4 of the theorem and the moments conditions (7) to (12) are standards.

(3) Conditions 3.a and 3.b are important for Itô's formula to be true in the case of fractional Brownian motion. Many stochastic processes can be chosen as $b(\tau)$. For example,

$$b(\tau) = A_1 \tau + A_2$$

where A_1 and A_2 are two random variables with $\mathbb{E} A_1^2 < \infty$ and A_1 is independent of $\{B_H(\tau)\}$.

Proof. Assume stochastic processes $b(\tau)$ and $\sigma(\tau)$ satisfy the conditions of Theorem 2.1. Then for any $t, s \in [0, T]$ such that $|t - s| \rightarrow 0$, we have

$$(2.16) \quad \int_s^t b(\tau) d\tau + \int_s^t \sigma(\tau) dB_H(\tau) = b(s)(t - s) + \sigma(s)(B_H(t) - B_H(s)) + 0_{L_2}(|t - s|),$$

where $0_{L_2}(|t - s|)$ means a term such that

$$(\mathbb{E} |0_{L_2}(|t - s|)|^2)^{\frac{1}{2}} = 0(|t - s|).$$

Here, $a(t, \omega)$ is a Riemann-Stieltjes integral, like $|t - s| \rightarrow 0$, by Lemma 16 of Dai and Heyde [3], we have

$$\int_s^t a(\tau) d\tau = a(s)(t - s) + 0_{L_2}(|t - s|).$$

Thus, to complete the proof of lemma, it suffices to show that

$$(2.17) \quad \int_s^t b(\tau) dB_H(\tau) = b(s)(B_H(t) - B_H(s)) + 0_{L_2}(|t - s|).$$

Without loss of generality, we assume $s < t$. Let be a partition sequence of $[s, t]$ in the form

$$\beta^{(n)} : s = t_0^{(n)} < t_1^{(n)} < \cdots < t_{q(n)}^{(n)} = t;$$

then

$$(2.18) \quad \mathbb{E} \left| \int_s^t b(\tau) dB_H(\tau) - b(s) (B_H(t) - B_H(s)) \right|^2 \\ = \lim_{n \rightarrow \infty} \mathbb{E} \left| \sum_{j=1}^n \left(b(t_{j-1}^{(n)}) - b(s) \right) \left(B_H(t_j^{(n)}) - B_H(t_{j-1}^{(n)}) \right) \right|^2.$$

Now Consider the term of the right hand side (2.18) without taking the limit yet. We have

$$(2.19) \quad \mathbb{E} \left| \sum_{j=1}^n \left(b(t_{j-1}^{(n)}) - b(s) \right) \left(B_H(t_j^{(n)}) - B_H(t_{j-1}^{(n)}) \right) \right|^2 \\ = \sum_{j,k=1}^n \mathbb{E} \left(\left(b(t_{j-1}^{(n)}) - b(s) \right) \left(B_H(t_{k-1}^{(n)}) - b(s) \right) \right) \\ \times \left(B_H(t_j^{(n)}) - B_H(t_{j-1}^{(n)}) \right) \left(B_H(t_k^{(n)}) - B_H(t_{k-1}^{(n)}) \right) \\ \equiv A_n + B_n.$$

If condition 1 of theorem 2.1 holds, then

$$(2.20) \quad A_n = \sum_{j=1}^n \mathbb{E} \left(b(t_{j-1}^{(n)}) - b(s) \right)^2 \mathbb{E} \left(B_H(t_j^{(n)}) - B_H(t_{j-1}^{(n)}) \right)^2 \\ \leq \text{const} |t - s|^{\beta+2H} = 0(|t - s|).$$

To treat B_n in (2.19) under the condition of theorem 3.1, we use the notation $\Gamma_{j,k}$, $\Delta\Gamma_{j,k}$ and α appearing in lemma 21 of Dai and Heyde [3]. $\left| \frac{\partial^2 \Gamma}{\partial y \partial x} \right|$ is integrable in $\{(x, y) : s \leq x \neq y \leq t\}$ by Lemma 21 of Dai and Heyde [3], equation (2.13) and the Cauchy-Schwartz inequality, we have

$$B_n = \sum_{j \neq k} \left\{ \mathbb{E} \left(b(t_{j-1}^{(n)}) - b(s) \right) \left(B_H(t_{k-1}^{(n)}) - b(s) \right) \right\} \Delta\Gamma_{j,k} \\ \leq \text{const} |t - s|^\beta \sum_{j \neq k} \left\{ \left| \frac{\partial^2 \Gamma}{\partial y \partial x} \left(t_{j-1}^{(n)}, t_{k-1}^{(n)} \right) \right| \left(t_j^{(n)} - t_{j-1}^{(n)} \right) \left(t_k^{(n)} - t_{k-1}^{(n)} \right) \right. \\ \left. + \left| t_{j-1}^{(n)} - t_{k-1}^{(n)} \right|^{2H-2-\alpha} 0 \left(\left(t_j^{(n)}, t_{j-1}^{(n)} \right) \left(t_k^{(n)} - t_{k-1}^{(n)} \right) \right) \right. \\ \left. + 0 \left(\left(t_j^{(n)}, t_{j-1}^{(n)} \right) \left(t_k^{(n)} - t_{k-1}^{(n)} \right) \right) \right\}$$

$$(2.21) \quad \rightarrow \text{const} |t - s|^\beta \int \int_{[s,t]^2} \left| \frac{\partial^2 \Gamma}{\partial y \partial x} \right| dy dx = 0(|t - s|).$$

Thus, in the case of condition 3.1, of (2.19), (2.20) and (2.21), the lemma holds. Finally, we consider the case of condition 3.1. By following (2.19) and using the inequality of condition 3.2, we have

$$(2.22) \quad \begin{aligned} A_n &= \sum_{j=1}^n E \left\{ \left(b' \left(t_{j-1}^{(n)} - s \right) + 0_{L_4} \left(t_{j-1}^{(n)} - s \right) \right)^2 \left(B_H(t_j^{(n)}) - B_H(t_{j-1}^{(n)}) \right)^2 \right\} \\ &\leq \text{const} (t - s)^2 \sum_{j=1}^{q(n)} \left(t_j^{(n)} - t_{j-1}^{(n)} \right)^{2H} = 0(t - s). \end{aligned}$$

By the same argument, under condition 3.2, we have

$$(2.23) \quad B_n \leq \text{const} (t - s)^2 \sum_{j \neq k} |\Delta \Gamma_{j,k}| = \text{const} |t - s|^{2+2H} = 0(t - s).$$

Thus, in the case of condition 3.2, of (2.19), (2.22) and (2.23), the lemma holds. This completes the proof of the lemma. $\square$

Theorem 2.2. *For any interval $[t_0, t] \subseteq [0, T]$ and any sequence of partitions $\beta^{(n)} : t_0 = t_0^{(n)} < t_1^{(n)} < \dots < t_{q(n)}^{(n)} = t$ with $|\beta^{(n)}| \rightarrow 0$ as $n \rightarrow \infty$, write $\Delta t_j^{(n)} = t_{j+1}^{(n)} - t_j^{(n)}$, $\Delta X_j^{(n)} = X_{t_{j+1}^{(n)}} - X_{t_j^{(n)}}$, $\Delta B_{H,j}^{(n)} = B_H(t_{j+1}^{(n)}) - B_H(t_j^{(n)})$, $\Delta U_j^{(n)} = U(t_{j+1}^{(n)}, X_{t_{j+1}^{(n)}}) - U(t_j^{(n)}, X_{t_j^{(n)}})$, for $j = 0, 1, \dots, q(n) - 1, n = 1, 2, \dots$. Then we have*

$$(2.24) \quad Y_t - Y_{t_0} = U(t, X_t) - U(t_0, X_{t_0}) = \lim_{n \rightarrow \infty} \sum_{i=0}^{q(n)} \Delta U_j^{(n)}.$$

Proof. From calculus, we have

$$(2.25) \quad \begin{aligned} \Delta U_j^{(n)} &= \frac{\partial U}{\partial t} \left(t_j^{(n)} + \theta_n \Delta t_j^{(n)}, X_{t_{j+1}^{(n)}} \right) \Delta t_j^{(n)} + \frac{\partial U}{\partial x} \left(t_j^{(n)}, X_{t_j^{(n)}} \right) \Delta X_j^{(n)} \\ &\quad + \frac{\partial^2 U}{\partial x^2} \left(t_j^{(n)}, X_{t_j^{(n)}} + \delta_n \Delta X_j^{(n)} \right) (\Delta X_j^{(n)})^2, \end{aligned}$$

where $\theta_n = \theta_n(\omega)$ and $\delta_n = \delta_n(\omega)$ are random variable such that $0 \leq \theta_n, \delta_n \leq 1$ and $\lim_{n \rightarrow \infty} \theta_n = \lim_{n \rightarrow \infty} \delta_n = 0$ in the $L_2(\Omega)$ sense. Since $\frac{\partial U}{\partial x}$ is uniformly continuous and the stochastic process X_t is continuous in the sense of $L_2(\Omega)$ (see theorem 16 Dai and Heyce [3]).

We have

$$(2.26) \quad \lim_{n \rightarrow \infty} \sum_{j=0}^{q(n)} \frac{\partial U}{\partial t} \left(t_j^{(n)} + \theta_n \Delta t_j^{(n)}, X_{t_{j+1}^{(n)}} \right) \Delta t_j^{(n)} = \int_{t_0}^t \frac{\partial U}{\partial t}(\tau, X_\tau) d\tau.$$

From Lemma we have

$$\Delta X_j^{(n)} = b(t_j^{(n)}) \Delta t_j^{(n)} + \sigma(t_j^{(n)}) \Delta B_{H,j}^{(n)} + 0_{L_2}(\Delta t_j^{(n)}),$$

where $0_{L_2}(\Delta t_j^{(n)})$ means a term such that

$$\left(\mathbb{E} |0_{L_2}(\Delta t_j^{(n)})|^2 \right)^{\frac{1}{2}} = 0 \left(\Delta t_j^{(n)} \right).$$

Therefore,

$$\begin{aligned} & \sum_{j=0}^{q(n)-1} \frac{\partial U}{\partial x}(t_j^{(n)}, X_{t_j^{(n)}}) \Delta X_j^{(n)} \\ &= \sum_{j=0}^{q(n)-1} \frac{\partial U}{\partial x}(t_j^{(n)}, X_{t_j^{(n)}}) \{b(t_j^{(n)}) \Delta t_j^{(n)} + \sigma(t_j^{(n)}) \Delta B_{H,j}^{(n)}\} + 0_{L_2}(1), \end{aligned}$$

where $0_{L_2}(1)$ means a term such that $\mathbb{E} |0_{L_2}(1)|^2 = 0$. Hence

$$(2.27) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \sum_{j=0}^{q(n)-1} \frac{\partial U}{\partial x} \left(t_j^{(n)}, X_{t_j^{(n)}} \right) \Delta X_j^{(n)} \\ &= \int_{t_0}^t \frac{\partial U}{\partial x}(\tau, X_\tau) \{b(\tau) + \sigma(\tau) dB_H(\tau)\}. \end{aligned}$$

from Lemma and noticing that

$$\mathbb{E} (B_H(t + \tau) - B_H(\tau))^2 = \tau^{2H} V_H$$

we have

$$\begin{aligned} & (\Delta X_j^{(n)})^2 = 0_{L_2}(\Delta t_j^{(n)}) \\ &= \frac{\partial^2 U}{\partial x^2}(t_j^{(n)}, X_{t_j^{(n)}} + \delta_n \Delta X_j^{(n)}) (\Delta X_j^{(n)})^2 = 0_{L_2}(\Delta t_j^{(n)}). \end{aligned}$$

and hence

$$(2.28) \quad \lim_{n \rightarrow \infty} \sum_{j=0}^{q(n)-1} \frac{\partial^2 U}{\partial x^2}(t_j^{(n)}, X_{t_j^{(n)}} + \delta_n \Delta X_j^{(n)}) (\Delta X_j^{(n)})^2 = 0.$$

Therefore, we have

$$Y_t - Y_{t_0} = \int_0^t \left\{ \frac{\partial U}{\partial t}(\tau, X_\tau) + b(\tau) \frac{\partial U}{\partial x}(\tau, X_\tau) \right\} d\tau + \int_0^t \frac{\partial U}{\partial x}(\tau, X_\tau) \sigma(\tau) dB_H(\tau),$$

or equivalent to

$$(2.29) \quad dY_t = \left\{ \frac{\partial U}{\partial t}(t, X_t) + b(t) \frac{\partial U}{\partial x}(t, X_t) \right\} dt + \sigma(t) \frac{\partial U}{\partial x}(t, X_t) dB_H(t).$$

□

3. APPLICATION TO THE STUDY OF SOME FRACTIONAL MODELS WITHOUT JUMP

We will propose some models of stochastic differential equation directed by a fractional Brownian motion without jumps.

3.1. Fractional Black-Scholes model. Here, we are interested in solving a stochastic differential equation of the fractional model without Black-Scholes jump defined below:

$$(3.1) \quad dX_t = bX_t dt + \sigma X_t dB_t^H,$$

where b and σ are constants and B_t^H is a fractional brownian motion of Hurst parameter $H \in (\frac{1}{2}, 1)$.

The previous equation cannot be solved within the framework of Itô's stochastic integral theory, because B_t^H is not a semi-martingale in general, except the case where $H = \frac{1}{2}$. New stochastic calculations are developed to deal with such.

Proof. To find the explicit solution of the equation, we apply Itô's formula with $Y_t = \ln(X_t)$

$$d(Y_t) = \frac{\partial F}{\partial t}(t, X_t) dt + \frac{\partial F}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(t, X_t) (dX_t)^2.$$

Let us $Y_t = F(t, X_t) = \ln(X_t)$ where F is a function with continuous derivatives up to order 2.

Indeed,

$$\frac{\partial F}{\partial t}(t, X_t) = 0, \quad \frac{\partial F}{\partial x}(t, X_t) = \frac{1}{X_t} \quad \text{and} \quad \frac{\partial^2 F}{\partial x^2}(t, X_t) = -\frac{1}{X_t^2},$$

or

$$d(Y_t) = d(\ln(X_t)) = \frac{dX_t}{X_t} - \frac{1}{2} \frac{d^2 X_t}{X_t^2},$$

or $dX_t = bX_t dt + \sigma X_t dB_t^H$, we have:

$$\begin{aligned} d(Y_t) &= \frac{bX_t dt + \sigma X_t dB_t^H}{X_t} - \frac{1}{2} \frac{(bX_t dt + \sigma X_t dB_t^H)^2}{X_t^2} \\ &= \frac{bX_t dt + \sigma X_t dB_t^H}{X_t} - \frac{1}{2} \frac{b^2 X_t^2 dt^2 + 2bX_t^2 \sigma dt \cdot dB_t^H + \sigma^2 X_t^2 (dB_t^H)^2}{X_t^2}. \end{aligned}$$

Using the following conventions

$$dt \cdot dB_t^H = dB_t^H \cdot dt = dt \cdot dt = 0 \text{ and } dB_t^H \cdot dB_t^H = \delta t$$

we have:

$$\begin{aligned} d(Y_t) &= \frac{bX_t dt + \sigma X_t dB_t^H}{X_t} - \frac{1}{2} \frac{\sigma^2 X_t^2 (dB_t^H)^2}{X_t^2} = bdt + \sigma dB_t^H - \frac{\sigma^2}{2} dt \\ d(Y_t) &= \left(b - \frac{\sigma^2}{2}\right) dt + \sigma dB_t^H \end{aligned}$$

or

$$F(t, X_t) = F(0, X_0) + \int_0^t \left(b - \frac{\sigma^2}{2}\right) ds + \sigma \int_0^t dB_s^H = Y_0 + \left(b - \frac{\sigma^2}{2}\right) t + \sigma B_t^H.$$

So,

$$(3.2) \quad X_t = X_0 \exp \left(\left(b - \frac{\sigma^2}{2}\right) t + \sigma B_t^H \right)$$

is unique solution of equation (3.1).

□

The conditional density function is log-nominal with the mean and the variance of its logarithmic transformation, i.e., the log-mean and la log-variance) given by

$$\mu = \log(X_0) + \left(b - \frac{\sigma^2}{2}\right) t, \quad \sigma_1^2 = \sigma^2 t$$

with mean and variance

$$\mathbb{E}[X_t] = \exp \left(\mu + \frac{1}{2} \sigma_1^2 t \right) = x_0 \exp(bt),$$

$$V[X_t] = \exp(2\mu + \sigma_1^2) (\exp(\sigma_1^2) - 1) = \left(\exp \left(\mu + \frac{1}{2} \sigma^2 t \right) \right)^2 (\exp(\sigma^2 t) - 1),$$

or $E[X_t] = \exp(\mu + \frac{1}{2}\sigma^2 t) = x_0 \exp(bt)$. So,

$$V[X_t] = (x_0 \exp(bt))^2 (\exp(\sigma^2 t) - 1) = x_0^2 \exp(2bt) (\exp(\sigma^2 t) - 1).$$

This allows us to say that the process X_t solution of the model converged in probability to 0 if $b < \frac{-\sigma^2}{2}$,

$$\begin{aligned} p_\theta(t, y | x_0) &= \frac{1}{\sigma_1 y \sqrt{2\pi}} \exp \left\{ -\frac{(\log y - \mu)^2}{2\sigma_1^2} \right\} \\ &= \frac{1}{\sigma y \sqrt{2\pi t}} \exp \left\{ -\frac{(\log y - (\log x + (b - \frac{1}{2}\sigma^2)t))^2}{2\sigma^2 t} \right\}. \end{aligned}$$

□

3.2. Fractional Ornstein-Uhlenbeck model. The fractional Ornstein-Uhlenbeck process (FOU) is a fractional analogue of the Ornstein-Uhlenbeck process. That is, a continuous process X that is the solution of the equation

$$(3.3) \quad dX_t = b(a - X_t)dt + \sigma X_t dB_t^H,$$

where $b > 0$, a and $\sigma > 0$ are parameters and $B^H = \{B_t^H\}_{t \geq 0}$ is a Fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. The solution of the equation (3.3) is:

$$(3.4) \quad X_t = X_0 e^{-bt} + a(1 - e^{-bt}) + \sigma \int_0^t e^{b(s-t)} dB_s^H.$$

Proof. The solution of the stochastic differential equation (3.3) can be found by applying the fractional Itô's lemma with $Y_t = e^{bt} X_t$,

$$d(Y_t) = \frac{\partial F}{\partial t}(t, X_t)dt + \frac{\partial F}{\partial x}(t, X_t)dX_t + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(t, X_t)(dX_t)^2.$$

Indeed, let $Y_t = F(t, X_t) = e^{bt} X_t$ where F a function with continuous derivatives up to order 2,

$$\frac{\partial F}{\partial t}(t, X_t) = b e^{bt} X_t = b F(t, X_t), \quad \frac{\partial F}{\partial x}(t, X_t) = e^{bt} \quad \text{et} \quad \frac{\partial^2 F}{\partial x^2}(t, X_t) = 0,$$

from where

$$d(Y_t) = d(e^{bt} X_t) = b e^{bt} X_t dt + e^{bt} dX_t.$$

Replace dX_t with its value in $d(Y_t)$,

$$d(X_t e^{bt}) = b e^{bt} X_t dt + a b e^{bt} dt - b e^{bt} X_t dt + \sigma e^{bt} dB_t^H = a b e^{bt} dt + \sigma e^{bt} dB_t^H$$

by integrating both sides, we have:

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t abe^{bs} ds + \int_0^t \sigma e^{bs} dB_s^H \\ &= X_0 + a(e^{bt} - 1) + \sigma \int_0^t e^{bs} dB_s^H. \end{aligned}$$

So,

$$e^{bt} X_t = X_0 + a(e^{bt} - 1) + \sigma \int_0^t e^{bs} dB_s^H,$$

and from here

$$X_t = X_0 e^{-bt} + a(1 - e^{-bt}) + \sigma \int_0^t e^{b(s-t)} dB_s^H,$$

where the stochastic integral is understood as a pathwise integral. $\square$

It follows that the process X of this model is a Gaussian process with a mean function

$$\mathbb{E}[X_t] = \mathbb{E}\left[X_0 e^{-bt} + a(1 - e^{-bt}) + \sigma \int_0^t e^{b(s-t)} dB_s^H\right].$$

Therefore,

$$\mathbb{E}[X_t] = X_0 e^{-bt} + a(1 - e^{-bt}),$$

and the covariance function

$$\begin{aligned} Cov(X_t, X_s) &= \mathbb{E}((X_s - \mathbb{E}(X_s))(X_t - \mathbb{E}(X_t))) \\ &= \sigma^2 e^{-b(s+t)} \int_0^s e^{2bu} du = \frac{\sigma^2}{2b} e^{-b(s+t)} (e^{2b \min\{t,s\}} - 1). \end{aligned}$$

and of variance

$$\begin{aligned} V[X_t] &= V\left(X_0 e^{-bt} + a(1 - e^{-bt}) + \sigma \int_0^t e^{b(s-t)} dB_s^H\right) \\ &= V(X_0 e^{-bt} + a(1 - e^{-bt})) + \sigma^2 V\left(\int_0^t e^{b(s-t)} dB_s^H\right). \end{aligned}$$

Thus, according to the isometry of $It\hat{o}$, we have:

$$\begin{aligned} V\left(\int_0^t e^{b(s-t)} dB_s^H\right) &= E\left[\left(\int_0^t e^{b(s-t)} dB_s^H\right)^2\right] = \mathbb{E}\left[\int_0^t e^{2b(s-t)} (dB_s^H)^2\right] \\ &= \mathbb{E}\left[\int_0^t e^{2b(s-t)} ds\right] = \frac{1}{2b}(1 - e^{-2bt}). \end{aligned}$$

Therefore

$$V[X_t] = \frac{\sigma^2}{2b}(1 - e^{2bt}).$$

This shows that the Ornstein-Uhlenbeck process converges in distribution to a Gaussian random variable of distribution $N(\mu, \frac{\sigma^2}{2b})$ when $t \rightarrow \infty$.

3.3. Fractional Lotka-Volterra model. The following Lotka-Volterra model equation

$$(3.5) \quad dX_t = X_t(a - bX_t)dt + \sigma X_t dB_t^H$$

has for solution

$$(3.6) \quad X_t = X_0 \exp\left(\left(a - \frac{\sigma^2}{2}\right)t + \sigma B_t^H\right) \left(1 + X_0 ab \int_0^t \Phi_s^{-1} ds\right)^{-1}$$

where $\Phi = e^{-(a - \frac{\sigma^2}{2})ht + \sigma B_t^H}$ (see [1])

Proof. The solution of the stochastic differential equation (3.5) is obtained by applying the Itô's formula to the transformation function $Y_t = F(t, X_t) = \ln X_t$, where F a function having continuous derivatives up to order 2 so that:

$$\frac{\partial F}{\partial t}(t, X_t) = 0, \quad \frac{\partial F}{\partial x}(t, X_t) = \frac{1}{X_t} \quad \text{and} \quad \frac{\partial^2 F}{\partial x^2}(t, X_t) = -\frac{1}{X_t^2}$$

or

$$d(Y_t) = d(\ln X_t) = X_t^{-1}dX_t - \frac{1}{2}X_t^{-2}(dX_t)^2.$$

Replacing dX_t of the stochastic differential equation above and we have:

$$\begin{aligned} d(Y_t) &= (X_t^{-1}X_t(a - bX_t)dt + \sigma X_t^{-1}X_t dB_t^H) \\ &\quad - \frac{1}{2}X_t^{-2}(X_t(a - bX_t)dt + \sigma X_t dB_t^H)^2 \\ &= ((a - bX_t)dt + \sigma dB_t^H) - \frac{1}{2}[(a - bX_t)^2 dt + \sigma^2 (dB_t^H)^2 \\ &\quad + 2a\sigma dB_t^H \cdot dt - 2b\sigma dt \cdot dB_t^H]. \end{aligned}$$

Using the following conventions

$$dt \cdot dB_t^H = dB_t^H \cdot dt = dt \cdot dt = 0 \quad \text{et} \quad dB_t^H \cdot dB_t^H = \delta dt,$$

we have:

$$(a - bX_t)^2 dt = 0, \quad \sigma^2 (dB_t^H)^2 = \sigma^2 dt, \quad 2a\sigma dB_t^H \cdot dt = 0 \quad \text{et} \quad 2b\sigma dt \cdot dB_t^H = 0$$

or

$$\begin{aligned} d(Y_t) &= (a - bX_t) dt + \sigma dB_t^H - \frac{1}{2}\sigma^2 dt \\ &= \left(a - bX_t - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t^H \\ Y_t &= Y_0 + \int_0^t \left(a - bX_s - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dB_s^H. \end{aligned}$$

Replacing Y_t by $\ln X_t$, we obtain

$$\ln X_t = Y_0 + \int_0^t \left(a - bX_s - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dB_s^H$$

which then gives

$$\begin{aligned} X_t &= \exp \left\{ Y_0 + \int_0^t \left(a - bX_s - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dB_s^H \right\} \\ &= \exp(Y_0) \exp \left(\int_0^t \left(a - bX_s - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dB_s^H \right). \end{aligned}$$

Therefore,

$$X_t = X_0 \exp \left(\left(a - \frac{\sigma^2}{2}\right)t + \sigma B_t^H \right) \left(1 + X_0 a b \int_0^t \Phi_s^{-1} ds \right)^{-1},$$

where $\Phi = e^{-[(a - \frac{\sigma^2}{2})t + \sigma B_t^H]}$. □

The last equation is a stochastic linear differential equation and it is solved using the previous formulas to give $Y_t = \ln X_t$.

3.4. Fractional Cox-Ingersoll-Ross fractionnaire(FCIR) model. The following Fractional Cox-Ingersoll-Ross model equation (FCIR)

$$(3.7) \quad dX_t = b(k - X_t) dt + \sigma \sqrt{X_t} dB_t^H,$$

where k , b and σ are constants. The stochastic differential equation has the explicit solution

$$(3.8) \quad X_t = e^{-bt} X_0 + k(1 - e^{-bt}) + \sigma \int_0^t e^{-b(t-s)} \sqrt{X_s} dB_s^H.$$

Proof. The solution of the stochastic differential equation (3.7) is obtained by applying the *Itô's* formula to the transformation function $Y_t = F(t, X_t) = e^{bt} X_t$, where F a function having continuous derivatives up to order 2 so that:

$$\frac{\partial F}{\partial t}(t, X_t) = be^{bt} X_t dt, \quad \frac{\partial F}{\partial x}(t, X_t) = e^{bt} \quad \text{and} \quad \frac{\partial^2 F}{\partial x^2}(t, X_t) = 0$$

or

$$d(Y_t) = d(e^{bt} X_t) = be^{bt} X_t dt + e^{bt} dX_t.$$

Replacing dX_t with its value in $d(Y_t)$,

$$\begin{aligned} d(Y_t) &= be^{bt} X_t dt + e^{bt} \left(b(k - X_t) dt + \sigma \sqrt{X_t} dB_t^H \right) \\ &= bke^{bt} dt + \sigma \sqrt{X_t} e^{bt} dB_t^H \end{aligned}$$

or

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t bke^{bs} ds + \int_0^t \sigma \sqrt{X_s} e^{bs} dB_s^H, \\ e^{bt} X_t &= X_0 + k(e^{bt} - 1) + \sigma \int_0^t \sqrt{X_s} e^{bs} dB_s^H. \end{aligned}$$

So,

$$X_t = X_0 e^{-bt} + k(1 - e^{-bt}) + \sigma \int_0^t \sqrt{X_s} e^{b(s-t)} dB_s^H.$$

□

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