J COMP SCI
APPL MATH
Journal of Computer Science and Applied Mathematics
Vol. 4, no.2, (2022), 51–63
ISSN: 1857-9582 https://doi.org/10.37418/jcsam.4.2.2

AN EXTENDED KATZ'S DISTRIBUTION OBTAINED BY THE BETA TRANSFORMATION FOR THE COUNT DATA

Michel Koukouatikissa Diafouka

ABSTRACT. In this paper, we introduce an extension of the Katz distribution constructed by the beta transformation. This is a new three-parameter distribution for the analysis and modeling of count data, which we call the new extended Katz distribution. We will study the new distribution from a probabilistic and statistical point of view. We perform a comparison study with an other extension of the Katz distribution with two methods: graphical and goodness-of-fit comparisons. For goodness-of-fit, we have considered the real data and the parameters are estimated by the maximum likelihood method.

1. INTRODUCTION

In count data modeling, the lack of adequacy of the reference model, in this case the Poisson model, leads to the formalization of one of the most important questions: how to formulate an adequate probability model to remedy the lack of adequacy of the Poisson model. This lack of adequacy is due to the variance of the sample being larger or smaller than the mean, because the Poisson model is only adequate if the variance is equal to the mean. These phenomena are called overdispersion, underdispersion, or equidispersion, respectively, for the variance that is larger, smaller, or equal to the mean. This is equivalent to the

Key words and phrases. Beta transformation, Katz's family, graphical comparison, goodness-of-fit.

Fisher dispersion index being greater, smaller, or equal to one, respectively. In the literature, several classes or families of probabilities have been proposed, taking into account overdispersion, equidispersion, and/or underdispersion. We can quote the weighted Poisson families coming from the concept of exponential families [13, 14] and the large Kemp class associated with generalized hypergeometric functions [6, 9]. In particular, the Katz family of distributions [8] as a special case of the last mentioned family, includes and generalizes the Poisson family of distributions with a variable Fisher dispersion index, which allows it to take into account situations of overdispersion, underdispersion, and equidispersion.

In [4], the authors have proposed a new specific transformation for discrete distributions called the beta transformation as one of the alternatives to the Poisson model. Given the distribution of a positive integer random variable, the beta transformation introduces an additional parameter. For a distribution of infinite support $\mathbb{N}$, we have the following definition [4]:

Definition 1.1. *Let X be a non-negative integer random variable with support $\mathbb{N}$ and the probability mass function (pmf), $p_k = P(X = k)$, $k \in \mathbb{N}$. The beta transformation of X is the non-negative integer random variable Y with support $\mathbb{N}$ and pmf, $p(k) = P(Y = k)$, given as follows:*

$$p(k) = \begin{cases} \dfrac{1-p_0}{\beta}, & k = 0, \\ p_{k-1} - \dfrac{p_k}{\beta}, & k = 1, 2, \ldots, \end{cases}$$

where β satisfies the conditions $\beta \geq 1 - p_0$ and $\beta \geq \max\limits_{k\geq 1} \left(\dfrac{p_k}{p_{k-1}}\right)$.

From the Definition (1.1), The main objective of this paper is to introduce the beta transformation of the Katz distribution as one of the alternatives to the Poisson distribution for modeling count data. Indeed, Katz [8] has formulated one of the best known probability models in the literature, whose recurrent ratio of probabilities is given by:

$$\frac{p_{k+1}}{p_k} = \frac{\lambda + \gamma k}{k+1}, \tag{1.1}$$

where $\lambda > 0$ and $\gamma < 1$, it is understood that if $\lambda + \gamma k < 0$ then $p_k = 0$ for $k = 1, 2, \ldots$ [2]. The pmf corresponding to (1.1) is given by (1.2):

$$p_k = \begin{cases} \dfrac{\lambda^k}{k!}e^{-\lambda}, & \text{if } \gamma = 0, \\ \dfrac{(\lambda/\gamma)_k \gamma^k}{k!}(1-\gamma)^{\lambda/\gamma}, & \text{otherwise}, \end{cases} \quad k = 0, 1, \ldots, \tag{1.2}$$

where $(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1)$ is the Pocchammer symbol, for $k = 0, 1, \ldots,$ and α any real number with $(\alpha)_0 = 1$. This family of distributions has been used as a basis to develop other families of distributions, such as the extensions of the Katz family proposed and developed in [1, 6, 15, 16].

The rest of the paper is as follows: in section 2, we present the beta transformation of the Katz distribution, which we call the new extension of the Katz distribution. In this section, we study, among other things, this new extension from a probabilistic and statistical point of view. In section 3, we carry out a comparison study with the Katz extension proposed in [6], an extension which has the same number of parameters as the new Katz extension, namely three parameters. We have chosen two methods of comparison: graphical comparison and by goodness-of-fit statistics. For the last comparison method, we considered real data and the parameters were estimated by the maximum likelihood method. Therefore, the asymptotic behavior of the estimators was not studied because it naturally follows from it. Finally, the work is closed with a conclusion in section 4.

2. New extended Katz's distribution

In this section, we successively discuss the following concepts in turn: probability mass function and recurrent ratio probabilities; probability generating function; moments and dispersion; and estimating of parameters.

2.1. **Probability mass function and recurrent ratio of probabilities.**

Let be X a Katz random variable with parameters (λ, γ) and suppose $\gamma \neq 0$, because, in this case, the Katz distribution reduces to the Poisson distribution with parameter λ:

Definition 2.1. *The pmf of the beta transformation Y of X, $p(k) = P(Y = k)$, is given by:*

$$p(k) = \begin{cases} \dfrac{1-(1-\gamma)^{\lambda/\gamma}}{\beta}, & k = 0, \\ \dfrac{(\lambda/\gamma)_k (1-\gamma)^{\lambda/\gamma}\gamma^k}{\beta k!}\left(\dfrac{\beta k}{\lambda+\gamma(k-1)} - 1\right), & k = 1, 2, \dots, \end{cases} \tag{2.1}$$

under conditions $\beta \geq 1-(1-\gamma)^{\lambda/\gamma}$ and $\beta \geq \max_{k\geq 1}\left(\dfrac{\lambda+\gamma(k-1)}{k}\right)$.

We call this beta transformation the new extended Katz distribution, denote $NEK(\lambda, \gamma, \beta)$.

It follows the following important remark:

Remark 2.1. *Note that for $\gamma < 0$, the support of the Katz distribution reduces to $\{0, 1, \dots, N\}$, where $N = -\dfrac{\lambda}{\gamma}$ or $N = \left[-\dfrac{\lambda}{\gamma}\right] + 1$ according as $-\dfrac{\lambda}{\gamma}$ is or is not an integer [15]. The beta transformation of this terminating distribution has as support $\{0, 1, \dots, N+1\}$ and the probability at $N+1$ is $\dfrac{(\lambda/\gamma)_N \gamma^N}{N!}(1-\gamma)^{\lambda/\gamma} = \left(\dfrac{-\gamma}{1-\gamma}\right)^N$, all else being equal. In this case, the Katz distribution is reduced to the binomial distribution and its beta transformation is called an extended binomial distribution in [4].*

Under the conditions on the parameters, (2.1) can be written as follows:

$$p(k) = \frac{(\lambda/\gamma)_k (1-\gamma)^{\lambda/\gamma}\gamma^k}{\beta k!}\left(\frac{\beta k}{\lambda+\gamma(k-1)} - 1\right) \times \left[\frac{1-(1-\gamma)^{\lambda/\gamma}}{\left(\dfrac{\beta k}{\lambda+\gamma(k-1)} - 1\right)(1-\gamma)^{\lambda/\gamma}}\right]^{\delta_0(k)}, \quad k = 0, 1, \dots \tag{2.2}$$

where δ_0 is the unit mass concentrated at zero. Relationshipship (2.2) is necessary to determine the maximum likelihood.

The recurrent ratio of probabilities corresponding to (2.1) is given by (2.3):

$$\frac{p(k+1)}{p(k)} = \begin{cases} \dfrac{(\beta-\lambda)(1-\gamma)^{\lambda/\gamma}}{1-(1-\gamma)^{\lambda/\gamma}}, & k=0 \\[2ex] \dfrac{[\gamma-\lambda+(\beta-\gamma)(k+1)][\lambda+\gamma(k-1)]}{[\gamma-\lambda+(\beta-\gamma)k](k+1)}, & k=1,2,\dots \end{cases} \tag{2.3}$$

In particular, if $\gamma = 0$, i.e. $(1-\gamma)^{\lambda/\gamma}|_{\gamma=0} = \lim_{\gamma\to 0}(1-\gamma)^{\lambda/\gamma} = e^{-\lambda}$, the $NEK(\lambda,\gamma,\beta)$ is the extended Poisson distribution, with parameters (λ,β), denote $EPo(\lambda,\beta)$ in [4].

2.2. **Probability generating function.** From [2] and Lemma 2.1 of [4], the probability generating function of $NEK(\lambda,\gamma,\beta)$ is given by (2.4):

$$G_Y(t) = \begin{cases} \dfrac{1-(1-\beta t)e^{\lambda(t-1)}}{\beta}, & \text{if } \gamma=0, \\[2ex] \dfrac{1}{\beta}\left[1-(1-\beta t)\left(\dfrac{1-\gamma t}{1-\gamma}\right)^{\lambda/\gamma}\right], & \text{otherwise}. \end{cases} \tag{2.4}$$

2.3. **Caracteristics: moments and dispersion.** From Corollary 2.1 of [4], the mean and the variance of $NEK(\lambda,\gamma,\beta)$ are given respectively by:

$$E(Y) = 1 + \frac{\lambda(1-\beta^{-1})}{1-\gamma},$$

and

$$V(Y) = \frac{\lambda(1-\beta^{-1}) + 2\lambda\beta^{-1}(1-\gamma) + \lambda^2(1-\beta^{-1})\beta^{-1}}{(1-\gamma)^2}.$$

For the dispersion, we have the Proposition 2.1.

Proposition 2.1. *The $NEK(\lambda,\gamma,\beta)$ family is:*

- *overdispersed if* $\lambda\left(1+\sqrt{\beta\left[1+\frac{\beta-1}{\lambda}\gamma\right]}\right) - \beta(1-\gamma) > 0$ *when* $\beta < 1$ *and* $\gamma < \frac{\lambda}{1-\beta}$ *or* $\beta > 1$ *and* $-\frac{\lambda}{\beta-1} < \gamma < 1$*;*
- *underdispersed if one of the following three conditions apply:*
  - $0 < \frac{\lambda}{1-\beta} < \gamma < 1$ *when* $\beta < 1$*;*

- $\gamma < -\dfrac{\lambda}{\beta-1} < 0$ *when* $\beta > 1$;
- $\lambda\left(1+\sqrt{\beta\left[1+\dfrac{\beta-1}{\lambda}\gamma\right]}\right) - \beta(1-\gamma) < 0$ *when* $\beta < 1$ *and* $\gamma < \dfrac{\lambda}{1-\beta}$ *or* $\beta > 1$ *and* $-\dfrac{\lambda}{\beta-1} < \gamma < 1$.

Proof. Note $I(Y)$ the Fisher dispersion index. We have:

$$V(Y) - E(Y) = \lambda^2\beta\left(1+\frac{\beta-1}{\lambda}\gamma\right) - [\lambda-\beta(1-\gamma)]^2 .$$

Thus, if $1+\dfrac{\beta-1}{\lambda}\gamma < 0$, i.e., $\beta < 1$ and $0 < \dfrac{\lambda}{1-\beta} < \gamma < 1$ or $\beta > 1$ and $\gamma < -\dfrac{\lambda}{\beta-1} < 0$, then $V(Y) - E(Y) < 0$, i.e., $I(Y) < 0$.

And if $1+\dfrac{\beta-1}{\lambda}\gamma > 0$, i.e., $\beta < 1$ and $\gamma < \dfrac{\lambda}{1-\beta}$ or $\beta > 1$ and $-\dfrac{\lambda}{\beta-1} < \gamma < 1$, we have:

$$\begin{aligned} V(Y) - E(Y) =& \frac{1}{\beta^2(1-\gamma)^2}\left[\lambda\sqrt{\beta\left[1+\frac{\beta-1}{\lambda}\gamma\right]} + \beta(1-\gamma) - \lambda\right] \times \\ & \left[\lambda\left(1+\sqrt{\beta\left[1+\frac{\beta-1}{\lambda}\gamma\right]}\right) - \beta(1-\gamma)\right] . \end{aligned} \tag{2.5}$$

The signe of $V(Y) - E(Y)$ depends only on the second factor of the second member of (2.5). Thus, $I(Y) > 1$ (< 1) if $\lambda\left(1+\sqrt{\beta\left[1+\dfrac{\beta-1}{\lambda}\gamma\right]}\right) - \beta(1-\gamma) > 0$ (< 0). □

In particular, if $\gamma = 0$, the Fisher dispersion index of the extended Poisson family is greater, smaller or equal to one according as $\lambda\left(1+\sqrt{\beta}\right) - \beta$ is positive, negative or null, respectively [3].

2.4. **Parameters estimation.** Let put $\theta = (\lambda, \gamma, \beta)$ the vector of parameters and consider a $n-$sample $x = x_1, \ldots, x_n$. Using the pmf of the new extended Katz distribution given by (2.2), the logarithm of the maximum likelihood function corresponding is:

$$l(\theta) = \frac{n\lambda}{\gamma}\left[1-\overline{\delta_0(x)}\right]\log(1-\gamma) + n\overline{\delta_0(x_i)}\log[1-(1-\gamma)^{\lambda/\gamma}] - n\log\beta + n\overline{x}\log\gamma + \sum_{i=1}^{n}\log(\lambda/\gamma)_{x_i} + \sum_{i=1}^{n}\log\left[\left(\frac{\beta x_i}{\lambda+\gamma(x_i-1)}-1\right)^{1-\delta_0(x_i)}\right] \quad (2.6)$$

where $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\overline{\log(x!)} = \frac{1}{n}\sum_{i=1}^{n}\log(x_i!)$ and $\overline{\delta_0(x_i)} = \frac{1}{n}\sum_{i=1}^{n}\delta_0(x_i)$. Relationship (2.6) is valid for $0 < \gamma < 1$, in particular. For $\gamma < 0$ or $\gamma = 0$, the log-likelihood is deduced from corresponding pmf (see [4] more details).

In practice, given (2.6), estimators can be easily determined using the maxLik package for the r statistical environment (see [7] for more details).

3. Comparison study with an other extension

In this section, we exploit two comparison methods: graphical and goodness-of-fit comparisons.

We compare the new extended Katz distribution with a three-parameter extended Katz distribution. Indeed, this three-parameter extension of the Katz distribution has been introduced in [6], denote EK, from the recurrent ratio of probabilities defined by:

$$\frac{p_{k+1}}{p_k} = \frac{\lambda+\gamma k}{k+\alpha}, \quad k = 0, 1, \dots, \quad (3.1)$$

where $\lambda > 0,\ \gamma < 1$ and $\alpha > 0$, it is always understood that for $\gamma < 0$ if $\lambda+\gamma k < 0$ then $p(k) = 0$ for each $k = 1, 2, \dots$ [2]. The pmf corresponding to (3.1) is [1]:

$$p_k = \frac{(\lambda/\gamma)_k(\gamma)^k}{(\alpha)_k}p_0, \quad k = 0, 1, \dots, \quad (3.2)$$

where $p_0^{-1} ={}_2 F_1(\lambda/\gamma, 1; \alpha; \gamma)$ with ${}_2F_1$ is the hypergeometric function. From (3.2), given an $n-$sample $x = (x_1, \dots, x_n)$ and putting $\theta = (\lambda, \gamma, \alpha)$ the vector of parameters, the log-likelihood maximum is gien by:

$$l(\theta) = \sum_{i=1}^{n}\log\left[(\lambda/\gamma)_{x_i}\gamma^{x_i}\right] - \sum_{i=1}^{n}\log(\alpha)_{x_i} - n\log\left[{}_2F_1(\lambda/\gamma, 1; \alpha; \gamma)\right].$$

For $\alpha = 1$, the EK family is reduced to the Katz family [6, 8].

3.1. **Graphical comparison.** We use the criterion $U_k = k\dfrac{p_k}{p_{k-1}}$ to compare the distributions graphically. This criterion was proposed in [12] and used in [16] to compare the EK family with an extension of the Crow-Bardwell family. To these families, we add the Poisson and Katz families.

For these distributions, U_k functions are:

(i) Poisson family: $U_k = \lambda,\ k = 1, 2, \ldots$
(ii) Katz family: $U_k = \lambda + \gamma(k-1),\ k = 1, 2, \ldots$
(iii) EK family: $U_k = \dfrac{k[\lambda + \gamma(k-1)]}{\alpha + k - 1},\ k = 1, 2, \ldots$
(iv) NEK family: $U_k = \begin{cases} \dfrac{(\beta-\lambda)(1-\gamma)^{\lambda/\gamma}}{1-(1-\gamma)^{\lambda/\gamma}}, & k = 1, \\ \dfrac{[\gamma - \lambda + (\beta-\gamma)k][\lambda + \gamma(k-2)]}{\gamma - \lambda + (\beta-\gamma)(k-1)}, & k = 2, 3, \ldots \end{cases}$

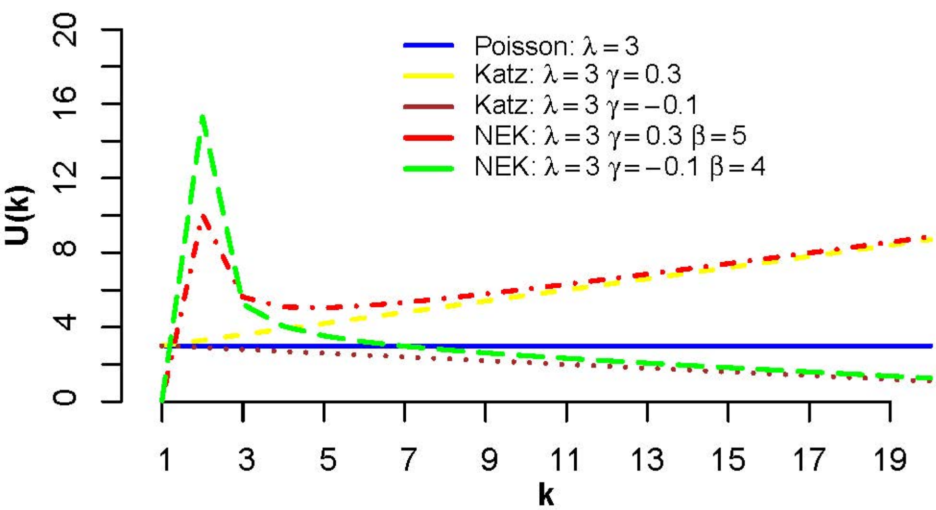

FIGURE 1. Comparison of K and NEK families

Figures 1 and 2 show the plots of U_k against k for different parameter values of these distributions. In Figure 1, we have plotted the U_k curves of the Katz distribution for $(\lambda = 3, \gamma = 0.3)$ and $(\lambda = 3, \gamma = -0.1)$ corresponding to the negative binomial and binomial distributions, respectively, as well as those of the new Katz extension for $(\lambda = 3, \gamma = 0.3, \beta = 5)$ and $(\lambda = 3, \gamma = -0.1, \beta = 4)$ corresponding to the extensions of the negative and binomial distributions, respectively. The NEK curves have a spike at the beginning (at point $k = 2$)

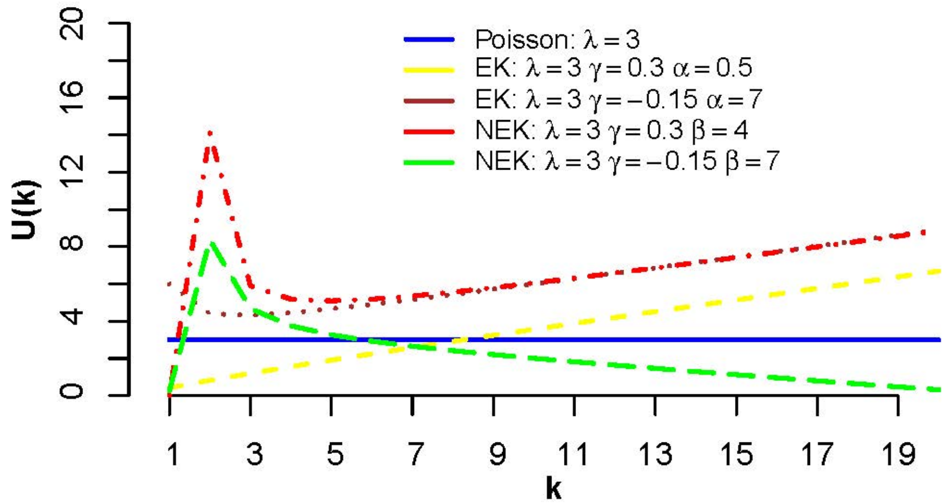

FIGURE 2. Comparison of EK and NEK families

and have the same behavior at the end as those of the Katz distribution. In Figure 2, we have plotted the U_k curves of the Katz extension for $(\lambda = 3, \gamma = -0.15, \alpha = 0.5)$ and $(\lambda = 3, \gamma = -0.15, \alpha = 7)$, and the new Katz extension for $(\lambda = 3, \gamma = -0.15, \beta = 4)$ and $(\lambda = 3, \gamma = -0.15, \beta = 7)$. The curves of the new extension always have a spike at the beginning (at point $k = 2$), while when $\alpha = 7$ for the Katz extension and $\beta = 4$ for the new extension, the curves have the same behavior at the end and are above the Poisson one. When $\alpha = 7$ for the Katz extension and $\beta = 7$ for the new extension, the curve of the Katz extension cuts and is above the Poisson one at the end while the one of the new extension cuts and is below the Poisson one.

3.2. **Goodness-of-fit comparison.** For the goodness-of-fit comparison, we use the following statistics: the Akaike's an information criterion (AIC) and the statistic of the Pearson's chi-squared test (χ^2). The $p - value$ and the log-likelihood value ($\log L$) are also presented.

We consider three types of real data: over-, equi- and under-dispersed data presented in Tables 1, 2 and 3, respectively. First data from in [5] and show the distribution of the number of accidents among 647 machine operators in a fixed period of time. In [5], these data were used to introduce the negative binomial distribution, a distribution that is considered the prototype of overdispersed distributions. The mean and variance of sample are 0.46522 and 0.6919, respectively. Second data from in [11] and show the distribution of spiders under boards; the mean and variance corresponding are 0.425 and 0.4546. Third data from in [10] and show the observed data on the number of outbreaks of strikes in 4-week periods, in a coal mining industry in the United Kingdom during 1948-1959. The mean and variance are 0.9935 and 0.7418, respectively.

TABLE 1. Number of accidents for machine operators [5]

Data		Distributions		
Count	Observed	Katz	EK	NEK
0	447	445.8969	446.9444	446.9877
1	132	134.8641	132.8266	131.0274
2	42	43.9947	44.6364	46.9515
3	21	14.7002	15.0002	15.1570
4	3	4.9701	5.0409	4.7534
5	2	2.5739	2.5514	2.1230
Total	467	646.9999	646.9999	647.0000
MLEs		$\widehat{\lambda} = 0.30246$	$\widehat{\lambda} = 0.00004542$	$\widehat{\lambda} = 0.32703$
		$\widehat{\gamma} = 0.34997$	$\widehat{\gamma} = 0.3361$	$\widehat{\gamma} = 0.30035$
			$\widehat{\alpha} = 0.0001528$	$\widehat{\beta} = 0.46637$
$\log L$		-592.2671	-592.1506	-506.175
AIC		1188.534	1190.301	1018.35
χ^2		3.7627	3.5061	3.4359
$p - value$		0.5841	0.6225	0.6331

For data in Tables 1 and 2, the new extended Katz family provides a better fit than the extended Katz for all goodness-of-fit statistics. In Table 3, the extended Katz family provides a better fit than the new extended Katz for the chi-square statistic; while the new extended Katz family has a better AIC than the extended Katz family. We have completed the Poisson family only in equidispersion situation, because we can't modeling the over- or under-dispersed data by an equidispersed distribution. As show in Table 2, in the equidispersion stuation, the Katz family fits slightly better than the extended Katz family.

TABLE 2. Distribution of spiders under boards [11]

Data		Distributions			
Count	Observed	Poisson	Katz	EK	NEK
0	159	156.9047	159.0621	159.1324	158.9950
1	64	66.6845	63.3431	63.1457	63.8050
2	13	14.1705	14.6105	14.7721	14.0583
3	4	2.0075	2.5539	2.5457	2.5929
4+	0	0.2328	0.4304	0.4041	0.5489
Total	240	240.0000	240.0000	240.0000	240.0001
MLEs		$\widehat{\lambda} = 0.4250$	$\widehat{\lambda} = 0.3982$	$\widehat{\lambda} = 0.48275$	$\widehat{\lambda} = 0.1909$
			$\widehat{\gamma} = 0.0631$	$\widehat{\gamma} = 0.0358$	$\widehat{\gamma} = 0.1623$
				$\widehat{\alpha} = 1.21656$	$\widehat{\beta} = 0.2838$
$\log L$		-205.4559	-205.2142	-205.2135	-189.0786
AIC		412.9118	414.4283	416.4269	384.1572
χ^2		2.4432	1.4336	1.4591	1.3928
$p-value$		0.6548	0.8383	0.8339	0.8455

TABLE 3. Number of outbreaks strikes [10]

Data		Distributions		
Count	Observed	Katz	EK	NEK
0	46	50.4395	46.0325	45.9786
1	76	65.0879	75.0405	75.3973
2	24	32.2716	26.7650	26.4167
3	9	7.4531	6.5470	6.4738
4	1	0.7478	1.6150	1.7336
Total	156	156.9999	156.0000	156.0000
MLEs		$\widehat{\lambda} = 1.2905$	$\widehat{\lambda} = 0.3142$	$\widehat{\lambda} = 0.3142$
		$\widehat{\gamma} = -0.2988$	$\widehat{\gamma} = 0.1111$	$\widehat{\gamma} = 0.1596$
			$\widehat{\alpha} = 0.1928$	$\widehat{\beta} = 0.9837$
$\log L$		-188.9164	-187.3729	-151.4964
AIC		381.8328	380.7459	308.9929
χ^2		4.7464	1.4513	1.5221
$p-value$		0.3143	0.8352	0.8227

4. CONCLUSION

The beta transformation is an attractive and interesting technique to construct the new variables as alternatives to the Poisson variable by adding an additional parameter to the distributions of the original variables. The new Katz extension constructed by this technique is as flexible and competitive as the three-parameter Katz extension, as shown by the numerical results in the comparative study. This new extension of the Katz family includes and generalizes, in particular, the extensions of the Poisson, binomial, and negative binomial families constructed by the beta transformation. Its Fisher dispersion index, which can be larger, smaller or equal to one, allows it to model both overdispersed, underdispersed equidispersion count data.

References

[1] M. Anwar, M. Ahmad: Characterization of extended Katz's (EK) family via negative moments, Pak. J. Statist., **28**(3) (2012), 375–384.

[2] M.A. Baradaran Ghahfarokhi, P. Baradaran Ghahfarokhi, Bahramian: *Application of katz family of distributions for detecting and testing overdispersion in Poisson models*, Journal of Statistical Modeling and Analytics, **1**(1) (2010), 29–44.

[3] R. Bidounga, M. Koukouatikissa Diafouka, R.F. Mizélé Kitoti, D. Mizère: *The Bivariate Extended Poisson Distribution of type 1*, Eur. J. Pure Appl. Math, **14**(4) (2021), 1517–1529.

[4] B. Dimitrov, N. Kolev: *Beta transformation. beta type self-decomposition and related characterizations*, Brezilian Journal of Probability and Statistics, **14** (2000), 123–140.

[5] M. Greenwood, U. Yule: *An inquiry into the nature of frequence distributions representative of multiple happenings with paritcular reference to the occurence of multiple attacks of dease or of repeated accidents*, J. Roy Statist. Soc. Ser. A, **83** (1920), 255–279.

[6] J. Gurland, R. Tripathi: *Estimation of parameters on some extensions of the Katz family of discrete distributions involving hypergeometric functions*, A modern course on statistical distributions in scientific work. Springer, Dordrecht, (1975), 59–82.

[7] A. Henningsen, O. Toomet: *maxlik: A package for maximum likelihood estimation in r*, Comput Stat, **26**(3) (2011), 443–458.

[8] L. Katz: *Unfied of a broad class of discrete probability distributions*, Classical and contagious discrete distributions, **1** (1965), 175–182.

[9] A.W. Kemp: *A wide class of discrete distributions and the associated differential equations*, Sankhyā: The Indian Journal of Statistics, Series A, (1968), 401–410.

[10] M.G. Kendall: *Natural law in the social sciences*, The inaugural address of the President (with discussion), OR (1961), 292–292.

[11] K.B. Kulasekera, D.W. Tonkyn: *A new discrete distribution, with applications to survival, dispersal and dispersion*, Communications in Statistics - Simulation and Computation, **21**(2) (1992), 499–518

[12] J. K. ORD: *Graphical methods for a class of discrete distributions*, Journal of the Royal Statistical Society: Series A (General) **130**(2) (1967), 232–238.

[13] G.P. PATIL, C.R. RAO: *Weighted distributions and size-biased sampling with applications to wildlife populations and human families*, Biometrics, **34**(2) (1978), 179–189.

[14] C.R. RAO: *On discrete distributions arising out of methods of ascertainment*, Sankhya Sr. A, 311–324.

[15] R.C. TRIPATHI, J. GURLAND: *A general family of discrete distributions with hypergeometric probabilities*, Journal of the Royal Statistical Society: Series B (Methodological), **39**(3) (1977), 349–356.

[16] R.C. TRIPATHI, J. GURLAND: *Some aspects of the kemp families of distributions*, Communications in statistics-theory and methods, **8**(9) (1979), 855–869.

HIGHER TEACHERS TRAINING COLLEGE
MARIEN NGOUABI UNIVERSITY
AVENUE DES PREMIERS JEUX AFRICAINS, BRAZZAVILLE,
CONGO.
Email address: michel.koukouatikissa@umng.cg