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HYBRID METHOD BASED ON EXPONENTIAL PENALTY FUNCTION AND MOMA-PLUS METHOD FOR MULTIOBJECTIVE OPTIMIZATION

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ABSTRACT. In this paper, we propose a modified version of the MOMA-plus method to solve multiobjective optimization problems. We use an exponential penalty function instead of the Lagrangian penalty function in the initial version of MOMA-Plus in order to improve the convergence and distribution to the Pareto optimal solutions. The theoretical and numerical results show that this new version improves the quality of the obtained solutions compared to the last version. Six test problems have been successfully resolved, allowing us to highlight the good convergence and good distribution of Pareto optimal solutions.

1. INTRODUCTION

Decision problems are mostly modeled as multiobjective optimization problems. Multiobjective optimization problems are mathematical representations in which multiple functions are optimized simultaneously, often with constraints. In general, multiobjective optimization problems do not have a unique solution due to the conflicting nature of objective functions. These problems are difficult to resolve, and there is no general method that is able to solve them efficiently.

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In the literature, there are many methods to resolve these kinds of problems. These methods can be classified into two main groups, namely the exact methods [7] and the metaheuristics [9, 17]. The methods of the first group are not suitable when the number of variables and/or objective functions is large. The second group of methods aims to find a good approximation of Pareto optimal solutions. Many methods proceed by transforming the multiobjective optimization problem into a single-objective optimization problem without constraint by using scalarization function and penalty function successively. That is the cas of MOMA-Plus method.

The MOMA-Plus method [21], is a method that transforms the initial problem into a single-objective optimization problem without constraints. It uses a Lagrangian penalty function and an aggregation function to convert any nonlinear multiobjective optimization problem. Weighted Chebychev distance is used in nonlinear cases and weighted son in linear cases. It is important to note that the MOMA-plus method has been used to solve several types of optimization problems, such as linear multiobjective optimization problems [19], nonlinear multiobjective optimization problems [20, 21], single-objective and multiobjective affectation and transport problems [14, 15], single-objective and multiobjective fuzzy optimization problems [2, 3], etc. Through these different works, MOMA-Plus has given satisfactory results, but it is not the best simultaneously on convergence, distribution and speed. This is why this current work aims to improve the performance of MOMA-Plus by using the exponential penalty function instead of the Lagrangian penalty function.

In the literature, there are many penalization techniques to convert a constrained problem into an unconstrained problem [5, 8, 10–12, 16, 22]. For this paper, we will focus on that of Sanming Liu et al [11] which propose a penalty function based on an exponential function. The optimality of solution of subproblem obtained bu using his exponential penalty function is proved. In addition, it is proved that, this penalty function is suitable for solving $\min - \max$ problem.

In this work, the coupling of MOMA-Plus method and the exponential penalty function has given a new method for the resolution of some multiobjective non-linear multiobjective optimization problems. We have demonstrated theoretically the convergence of the algorithm of the new version. In addition, six test problems [6] are solved and numerical solutions are compared to those of the last version. This allowed us to highlight the performance of the novel version.

For a best presentation of this work, Section 2 will be used as preliminary. Section 3 will include the main results of this paper, and Section 4 will deal with the conclusion.

2. Preliminaries

2.1. **Definitions.** A multiobjective optimization problem can be described as follows:

$$\begin{array}{ll} \min & f(x) = (f_1(x), f_2(x), \dots, f_p(x)) \\ s.t: & \begin{cases} g(x) \leq 0 \\ x \in \mathbb{R}^n \end{cases} \end{array} \tag{2.1}$$

where $f = (f_1, f_2, \dots, f_p)$ is the vector whose components are the objectives functions and $g = (g_1, g_2, \dots, g_m)$ is the vector whose components are the constraints functions.

For the solution of such a problem, it is necessary to know some sets as the decision space (noted by χ) and objective space (noted by $\mathcal{Y}$) with are respectively the set of admissible solutions and its image by f. We have

$$\chi = \{x \in \mathbb{R}^n : \ g(x) \leq 0\} \text{ and } \mathcal{Y} = \{f(x) : x \in \chi\}.$$

Definition 2.1. *A solution $x^* \in \chi$ is weakly Pareto optimal for the problem (2.1) if and only if there is no point $x \in \chi$ such that:*

$$f_j(x) < f_j(x^*), \quad \forall j = 1, 2, \dots, p.$$

Definition 2.2. *A solution $x^* \in \chi$ is Pareto optimal if there is no solution $x \in \chi$ such that $f_j(x) \leq f_j(x^*)$, $\forall j = 1, 2, \dots, p$ and for at least one $k \in \{1, 2, \dots, p\}$, we have $f_k(x) < f_k(x^*)$.*

In addition to these definitions, we will also use the notion of ideal point.

Definition 2.3 ([7]). *The ideal point is the vector $z^* \in \mathbb{R}^p$ whose components z_j^* are obtained by individually minimizing each objective function $f_j,\ j = 1, 2, \dots, p$ under the constraints of the initial multiobjective optimization problem. In other words*

$$z_j^* = \min\ f_j(x) \\ s.t:\ \begin{cases} g(x) \le 0 \\ x \in \mathbb{R}^n. \end{cases} \tag{2.2}$$

Definition 2.4. *The weighted Chebychev distance is a function which allows to transform multiobjective optimization function to single-objective optimization function. It is formulates by:*

$$\Psi(f(x), \lambda, z^*) = \max_{j=1,2,\dots,p} \Big\{\lambda_j |f_j(x) - z_j^*|\Big\}, \tag{2.3}$$

with $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ such that $\lambda_j > 0$ and $\sum_j^p \lambda_j = 1$.

2.2. **Exponential penalization function.** A Penalty function is a function that transform a constrained optimization problem into an unconstrained optimization problem. In this work, we will only use the exponential penalization defined in [11].

Definition 2.5. *([11]) The exponential penalty function is defined as follows:*

$$\Pi_n(x) = \frac{1}{\kappa_n} \sum_{i=1}^m \upsilon[\kappa_n g_i(x)], \tag{2.4}$$

where κ_n is the penalty coefficient verifying the property:

$$\lim_{n \to +\infty} \kappa_n = +\infty, \tag{2.5}$$

and υ is a real-valued function defined by:

$$\upsilon(t) = \exp(t - 1). \tag{2.6}$$

Using the penalty function (2.4) allows us to transform the problem (2.1) into a multiobjective optimization problem without constraints. In other words:

$$\min \quad \Big\{f(x) + \frac{1}{\kappa_n} \sum_{i=1}^m \upsilon[\kappa_n g_i(x)]\Big\}. \tag{2.7}$$

This is equivalent to:

$$\min \quad \Big\{f_1(x)+\frac{1}{\kappa_n}\sum_{i=1}^{m}\upsilon[\kappa_n g_i(x)],\ldots,f_p(x)+\frac{1}{\kappa_n}\sum_{i=1}^{m}\upsilon[\kappa_n g_i(x)]\Big\}. \tag{2.8}$$

Definition 2.6. *([11]) Let $S_n \subset \mathbb{R}^k$, $n \in \{1,2,\ldots\}$, denote:*

- ✓ $\overline{\lim\limits_{n\to+\infty}}\, S_n = \{x \in \mathbb{R}^k : \ x \in S_n, \ $ *for infinitely many* $n \in \mathbb{N}\}$*;*
- ✓ $\underline{\lim\limits_{n\to+\infty}}\, S_n = \{x \in \mathbb{R}^k : \ x \in S_n, \ $ *for all but finitely many* $n \in \mathbb{N}\}$.

Note that χ^* is the set of weakly Pareto optimal solutions of the problem (2.1); χ_n^* is the set of weakly Pareto optimal solutions of the problem (2.8); Ω^* is the the set of Pareto optimal solutions of the problem (2.1) and Ω_n^* the the set of Pareto optimal solutions of the problem (2.8) .

2.3. **Alienor transformation.**

Definition 2.7. *[3] We call Alienor transformation, any transformation allowing to reduce a function of several variables to a function of single variable with the help of $\alpha-$dense curves.*

$\alpha-$dense curves are studied in [4] and the interested reader can consult it. The Alienor transformation that we use in this paper is the Konfe-Cherruault transformation [1]. It is given by the following relation (2.9):

$$x_i=\sigma_i(\theta)=\frac{1}{2}\Big[(b_i-a_i)\cos(\omega_i\theta+\phi_i)+a_i+b_i\Big],\ i=\{1,\ldots,n\} \tag{2.9}$$

where ω_i and ϕ_i are slowly increasing sequences and $\theta \in [0;\theta_{max}]$, with
$\theta_{max}=\dfrac{(b-a)\theta^1+(b+a)}{2}$ and $\theta^1=\dfrac{2\pi-\phi_1}{\omega_1}$, with $a=\min\limits_{i=\overline{1,n}} a_i$ and $b=\max\limits_{i=\overline{1,n}} b_i$.

Theorem 2.1. *[3] Any point $x_i \in [a_i;b_i]$ can be approximated by at least one point defined by the Alienor transformation $\sigma_i(\theta)$.*

2.4. **MOMA-plus method.** MOMA-plus method was developed by Somé. K. et al [20] for solving multiobjective optimization problems. The different steps of the MOMA-plus method are :

1. **STEP I: Scalarization** : transform the problem (2.1) into a single-objective optimization problem with constraints.
2. **STEP II: Penalization** : transform the last formulation into a single-objective optimization problem without constraints.

(3) **STEP III: Alienor transformation** : transform the obtained formulation at the step II into one variable optimization problem;
(4) **STEP IV: Solutions research** : resolve the single-objective optimization problem with one variable;
(5) **STEP V: Solution initialization** : transform the obtained solution in $\mathbb{R}$ to the solution of the initial problem.

3. Main results

3.1. Theoretical results. In this work, MOMA-Plus method is modified at two levels. At the one hand we start this version by the penalization and at the other hand we replace the Lagrangian penalty function by the exponential penalty function.

3.1.1. *Penalization of objective functions.* At this step, we have used the the exponential penalty function to transform constrained optimization problem into unconstrained optimization problem.

Theorem 3.1. *All Pareto optimal of (2.8) is also Pareto optimal solution of (2.1) and reciprocally.*

Proof. Let x^* be a Pareto optimal solution of problem (2.8). Suppose that x^* is not a Pareto optimal solution of problem (2.1) then there exist $y \in \chi$ such that $\forall\, j = 1, \ldots, p$, $f_j(y) \leq f_j(x^*)$ and at last one $k \in \{1, \ldots, p\}$, $f_k(y) < f_k(x^*)$. As $x^*,\ y \in \chi$ then there exist $n_0 \geq 0$ such that $\forall n \geq n_0$,

$$f_j(y) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(y)] \leq f_j(x^*) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(x^*)],$$

$\forall j = \{1, \ldots, p\}$, and all $k \in \{1, \ldots, p\}$. This implies

$$f_k(y) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(y)] < f_k(x^*) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(x^*)].$$

That is absurd because x^* is a Pareto optimal solution of problem (2.8).

Conversly, let x^* be a Pareto optimal solution of problem (2.1). Suppose that x^* is not a Pareto optimal solution of problem (2.8), then there exist $x \in \chi$ such that

$$_j(x) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(x)] \leq f_j(x^*) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(x^*)],$$

$\forall j = 1, \dots, p$ and at last one $k \in \{1, \dots, p\}$,

$$f_k(x) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(x)] < f_k(x^*) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(x^*)].$$

As $x,\ x^* \in \chi$, there exist $n_0 \geq 0$ such that $\forall n \geq n_0$, $f_j(x) \leq f_j(x^*)$, $\forall j = 1 \dots, p$ and at last one $k \in \{1, \dots, p\}$, $f_k(x) < f_k(x^*)$. That is absurd because x^* is a Pareto optimal solution of problem (2.1). □

3.1.2. *Scalarization.* To transform the problem (2.1) into an unconstrained single-objective optimization problem we propose to use the relation (2.8) and to apply an aggregation technique. By replacing f_j of the relation (2.3) by:

$$F_j(x) = f_j(x) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(x)], \tag{3.1}$$

we obtain the aggregation function Ψ written as:

$$\Psi(F, \lambda, z^*) = \max_{j=1,2,\dots,p} \left[\lambda_j |F_j - z_j^*|\right]. \tag{3.2}$$

The problem (2.8) becomes:

$$\min_x\ \Psi(F(x), \lambda, z^*). \tag{3.3}$$

This transformation gives us a single objective optimization problem without constraints.

Let x^* be the optimal solution of the problem (3.3) for each $(\lambda_1, \lambda_2, \dots, \lambda_p) > 0$ such that $\lambda_1 + \lambda_2 + \dots + \lambda_p = 1$ and let Y_n be the set of all optimal solutions of the problem (3.3). We have the following theorems :

Theorem 3.2. $\overline{\lim\limits_{n\to+\infty}} (Y_n \backslash \chi_n^*) = \emptyset.$

Proof. Assume that $\overline{\lim\limits_{n\to+\infty}} (Y_n \backslash \chi_n^*) \neq \emptyset$, then $\exists x^* \in \overline{\lim\limits_{n\to+\infty}} (Y_n \backslash \chi_n^*)$ and a subset $n_\rho \subset \mathbb{N}$, $\rho \in \{1, 2, \dots\}$ such as $x^* \in (Y_{n_\rho} \backslash \chi_{n_\rho}^*)$. Then $x^* \in Y_{n_\rho}$ and $x^* \notin \chi_{n_\rho}^*$. Since $x^* \notin \chi_{n_\rho}^*$, then

$$\exists y \in \chi_{n_\rho}^* : \quad F_j(y) < F_j(x^*) \quad , \forall j = 1, \dots, p. \tag{3.4}$$

If $x^* \in \chi$, $\exists x' \in \chi^*_{n_\rho} \quad F_k(x') < F_k(x^*) \quad \forall k = \{1, 2, \ldots, p\}$

$\implies F_k(x') - z^*_k < F_k(x^*) - z^*_k$.
$\implies |F_k(x') - z^*_k| < |F_k(x^*) - z^*_k|$ because $F_k(x') - z^*_k > 0,\ \forall k = 1, 2, \ldots, p$.
$\implies \lambda_k |F_k(x') - z^*_k| < \lambda_k |F_k(x^*) - z^*_k|$ because $\lambda_j > 0,\ \forall j = 1, 2, \ldots, p.$
$\implies \max\limits_{k=1,2,\ldots,p} \{\lambda_k |F_k(x') - z^*_k|\} < \max\limits_{k=1,2,\ldots,p} \{\lambda_k |F_k(x^*) - z^*_k|\}.$

That is equivalent to $\Psi(F(x'), \lambda, z^*) < \Psi(F(x^*), \lambda, z^*)$; which is absurd because x^* is the optimal solution of the problem (3.3).

If $x^* \notin \chi$, then there is $y \in \chi$, such that $f_j(y) < f_j(x^*)\ \forall \quad j = 1, 2, \ldots, p$. As $y \in \chi$ and $x^* \notin \chi$, we have:

$$\lim_{n_\rho \longrightarrow +\infty} \frac{1}{\kappa_{n_\rho}} \sum_{i=1}^{m} \upsilon[\kappa_{n_\rho} g_i(y)] = 0 \text{ and } \lim_{n_\rho \longrightarrow +\infty} \frac{1}{\kappa_{n_\rho}} \sum_{i=1}^{m} \upsilon[\kappa_{n_\rho} g_i(x^*)] = +\infty.$$

Then, there exist $\rho_0 \in \mathbb{N}$, such that for $\rho \geq \rho_0$:

$$f_j(y) + \frac{1}{\kappa_{n_\rho}} \sum_{i=1}^{m} \upsilon[\kappa_{n_\rho} g_i(y)] < f_j(x^*) + \frac{1}{\kappa_{n_\rho}} \sum_{i=1}^{m} \upsilon[\kappa_{n_\rho} g_i(x^*)].$$

Hence: $F_j(y) < F_j(x^*)$,

$\implies F_j(y) - z^*_j < F_j(x^*) - z^*_j, \forall j = 1, 2, \ldots, p.$
$\implies \lambda_j |F_j(y) - z^*| < \lambda_j |F_j(x^*) - z^*|, \forall\, j = 1, 2, \ldots, p.$

Hence $\max\limits_{j=1,\ldots,p} \{\lambda_j |F_j(y) - z^*|\} < \max\limits_{j=1,\ldots,p} \{\lambda_j |F_j(x^*) - z^*|\}.$
$\Leftrightarrow \Psi(F(y), \lambda, z^*) < \Psi(F(x^*), \lambda, z^*)$. Which is absurd, then $\overline{\lim\limits_{n \to +\infty}} (Y_n \backslash \chi^*_n) = \emptyset$ $\square$

Theorem 3.3. $\underline{\lim\limits_{n \to +\infty}} (Y_n \backslash \chi^*_n) = \emptyset.$

Proof. Let's assume that $\underline{\lim\limits_{n \to +\infty}} (Y_n \backslash \chi^*_n) \neq \emptyset$, then $\exists x^* \in \underline{\lim\limits_{n \to +\infty}} (Y_n \backslash \chi^*_n)$. $\exists n_0 \in \mathbb{N}$, such as $\forall\, n \geq n_0$: $x^* \in Y_n \backslash \chi^*_n$. Then $x^* \in Y_n$ and $x^* \notin \chi^*_n$.

As $x^* \notin \chi^*_n$, then

(3.5) $$\exists y \in \chi^*_n : \quad F_j(y) < F_j(x^*), \forall j = 1, 2, \ldots, p.$$

If $x^* \in \chi$, $\exists x' \in \chi^*_n \quad et \quad k = 1, 2, \ldots, p$ such that $F_k(x') < F_k(x^*)$.

$\implies \exists x' \in \chi^*_n \quad F_k(x') - z^*_k < F_k(x^*) - z^*_k,\ \forall k = 1, 2, \ldots, p.$
$\implies |F_k(x') - z^*_k| < |F_k(x^*) - z^*_k|$, because $F_j(x') - z^*_j > 0,\ \forall j = 1, 2, \ldots, p.$

As $\lambda_j > 0,\ \lambda_k|F_k(x') - z_k^*| < \lambda_k|F_k(x^*) - z_k^*|, \forall j = 1, 2, \ldots, p$. We have

$$\max_{k=1,\ldots,p}\{\lambda_k|F_k(x') - z_k^*|\} < \max_{k=1,\ldots,p}\{\lambda_k|F_k(x^*) - z_k^*|\}.$$

That is equivalent to $\Psi(F(x'), \lambda, z^*) < \Psi(F(x^*), \lambda, z^*)$, which is absurd because x^* is the optimal solution of the problem (3.3).

If $x^* \notin \chi$, then there exist $y \in \chi$ such that $f_j(y) < f_j(x^*)$, $\forall j = 1, 2, \ldots, p$. Then

$$f_j(y) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(y)] < f_j(x^*) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(x^*)].$$

This is equivalent to: $F_j(y) < F_j(x^*), \forall j = 1, 2, \ldots, p$. So $F_j(y) - z_j^* < F_j(x^*) - z_j^*, \forall j = 1, 2, \ldots, p$.

$\Longrightarrow \lambda_j|F_j(y) - z^*| < \lambda_j|F_j(x^*) - z^*|$, $\forall\, j = 1, 2, \ldots, p$.
$\Longrightarrow \max_{j=1,2,\ldots,p}\{\lambda_j|F_j(y) - z^*|\} < \max_{j=1,2,\ldots,p}\{\lambda_j|F_j(x^*) - z^*|\}$.
$\Longrightarrow \Psi(F(y), \lambda, z^*) < \Psi(F(x^*), \lambda, z^*)$. That is absurd, because $x^* \in Y_n$.

Hence $\underline{\lim}_{n\to+\infty} (Y_n \backslash \chi_n^*) = \emptyset$ $\square$

Theorem 3.4. *Any optimal solution of problem (3.3) is a Pareto optimal solution of problem (2.8) and reciprocally.*

Proof. Let x^* be the optimal solution of problem (3.3), then $\forall\, y \in \chi$; $\Psi(F(x^*), \lambda, z^*) < \Psi(F(y), \lambda, z^*)$. Suppose that x^* is not a Pareto optimal solution of (2.8), then $\exists x \in \chi$ such that

$$f_j(x) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(x)] \le f_j(x^*) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(x^*)],$$

$\forall j = 1, 2, \ldots, p$ and at last one $k \in \{1, \ldots, p\}$ such that

$$f_k(x) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(x)] < f_k(x^*) + \frac{1}{\kappa_n}\sum_{i=1}^{m} \upsilon[\kappa_n g_i(x^*)].$$

$\Rightarrow F_j(x) \le F_j(x^*),\ j = 1, 2, \ldots, p$ and at last one $k \in \{1, \ldots, p\}$ we have $F_k(x) \le F_k(x^*)$.

As z^* is the ideal point of (2.8), then

$\Longrightarrow F_j(x) - z_j^* \le F_j(x^*) - z_j^*, \forall j = 1, 2, \ldots, p$ and at last one $k = \{1, \ldots, p\}$, $F_k(x) - z_k^* < F_k(x^*) - z_k^*$.

$\Longrightarrow$ $|F_j(x)-z_j^*| \leq |F_j(x^*)-z_j^*|, \forall j = 1,2,\ldots,p$ and at last one $k \in \{1,\ldots,p\}$, $|F_k(x)-z_k^*| < |F_k(x^*)-z_k^*|$.
$\Longrightarrow$ $\lambda_j|F_j(x)-z_j^*| \leq \lambda_j|F_j(x^*)-z_j^*|, \forall j = 1,2,\ldots,p$ and at last one $k \in \{1,\ldots,p\}$, such that
$\lambda_k|F_k(x)-z_k^*| < \lambda_k|F_k(x^*)-z_k^*|$, because $\lambda_l > 0,\ \forall l = 1,2,\ldots,p.$
$\Longrightarrow$ $\exists x \in \chi$ such that $\Psi(F(x),\lambda,z^*) \leq \Psi(F(x^*),\lambda,z^*)$.

That is absurd because x^* is an optimal solution for the problem (3.3).
Conversly, let $\overline{x}$ be a Pareto optimal solution of problem (2.8). Suppose that $\overline{x}$ is not an optimal solution of problem (3.3), then exist $x \in \chi$ such that $\Psi(F(x),\lambda,z^*) < \Psi(F(\overline{x}),\lambda,z^*)$. As $\overline{x}$ is a Pareto optimal solution of (2.8), $\nexists y \in \chi$, such that $f_j(y) + \dfrac{1}{\kappa_n}\sum\limits_{i=1}^{m} \upsilon[\kappa_n g_i(y)] - z_j^* \leq f_j(\overline{x}) + \dfrac{1}{\kappa_n}\sum\limits_{i=1}^{m} \upsilon[\kappa_n g_i(\overline{x})] - z_j^*$, $\forall j = 1,2,\ldots,p$ and and last one $k \in \{1,\ldots,p\}$ we have, $f_k(y) + \dfrac{1}{\kappa_n}\sum\limits_{i=1}^{m} \upsilon[\kappa_n g_i(y)] - z_k^* <$ $f_k(\overline{x}) + \dfrac{1}{\kappa_n}\sum\limits_{i=1}^{m} \upsilon[\kappa_n g_i(\overline{x})] - z_k^*$,
Then $F_j(y) - z_j^* \leq F_j(\overline{x}) - z_j^*$, $\forall j = 1,2,\ldots,p$ and at last one $k \in \{1,\ldots,p\}$, $F_k(y) - z_k^* < F_k(\overline{x}) - z_k^*$.
So $\nexists\, y \in \chi,\ \lambda_j|F_j(y)-z_j^*| \leq \lambda_j|F_j(\overline{x})-z_j^*|,\ \forall j = 1,2,\ldots,p$ and at last one $k \in \{1,\ldots,p\}$ $\lambda_k|F_k(y)-z_k^*| < \lambda_k|F_k(\overline{x})-z_k^*|$.
Then $\nexists\, y \in \chi$ such that $\Psi(F(y),\lambda,z) < \Psi(F(\overline{x}),\lambda,z)$. Therefore $\overline{x}$ is the optimal solution of problem (3.3). □

3.1.3. *Alienor Transformation [3].* The application of the relation (2.9) to the problem (3.3), give us a single-objective optimization problem of only one variable represented by the relation (3.6):

$$\begin{cases} \min\ L(\theta) \\ \theta \in [0; \theta_{max}] \end{cases} \tag{3.6}$$

Theorem 3.5. *([3]) Any minimum of the problem (3.3) can be approximated by a minimum of the problem (3.6).*

Proof. For the proof, see in [3]. □

3.1.4. *Solutions research.* Since the problem (3.6) is single-objective with only one variable then we will use the Nelder-Mead algorithm [13] which is appropriate for its solutions.

3.1.5. *Solutions initialization.* The obtained solution at the previous step must be transform to the solution of the initial problem. That is possible by re-using of the Alienor transformation described in Section 2.3.

3.1.6. *Modified algorithm of MOMA-plus.* The main lines of the modified MOMA-plus method are defined as follows :

Input: Enter the value of κ_n and choose λ_j such as $\sum_{j=1}^{p} \lambda_j = 1$

$F(x) \longleftarrow \left\{ f(x) + \frac{1}{\kappa_n} \sum_{i=1}^{m} \upsilon[\kappa_n g_i(x)] \right\}$

for *For* $j \leftarrow 1$ **to** p **do**

  $\Psi(F(x), \lambda, z^*) \leftarrow \max \left[\lambda_j |F_j(x) - z_j^*| \right]$

  $\Gamma(x) \leftarrow \Psi(F(x), \lambda, z^*)$

end

for $i \leftarrow 1$ **to** n **do**

  $\sigma_i(\theta) \leftarrow x_i$

end

$\Gamma(\theta) \leftarrow \Gamma(\sigma_1(\theta), \sigma_2(\theta), \ldots, \sigma_n(\theta))$

return θ^* found by Nelder-Mead algorithm [13]

for $i \leftarrow 1$ **to** n **do**

  $x_i \longleftarrow \sigma(\theta^*)$

end

return Display the solution x of the problem which is one of the best compromise corresponding to fixed λ_k;

Algorithm 1: MOMA-PLUS MODIFIED ALGORITHM

3.2. **Numerical results.**

3.2.1. *Presentation of the test problems.* The multiobjective optimization problems we will study are the Zitzler test problems [6] and are listed in the table 1.

TABLE 1. Multiobjective problems

Indexes	Multiobjective problems	n	Bounds
T_1	$\begin{cases} minf_1(x_1,x_2) = x_1 \\ minf_2(x_1,x_2) = \dfrac{1+x_2}{x_1} \\ 0.1 \leq x_1 \leq 1 \\ 0 \leq x_2 \leq 5 \end{cases}$	2	$x_1, x_2 \in [0;1]$
T_2	$\begin{cases} minf_1(x) = x^2 \\ minf_2(x) = (x-2)^2 \\ -5 \leq x \leq 5 \end{cases}$	1	$x \in [0;4]$
T_3	$\begin{cases} minf_1(x) = x_1 \\ minf_2(x) = g\left(1 - \sqrt{\frac{f_1(x)}{g(x)}}\right) \\ g(x) = 1 + \dfrac{9}{n-1} \times \sum\limits_{i=2}^{n} x_i \\ x = (x_1, x_2, \ldots, x_n) \in [0.1]^n \end{cases}$	30	$x_i \in [0;1]$
T_4	$\begin{cases} minf_1(x) = x_1 \\ minf_2(x) = g(x)\left(1 - \left(\dfrac{f_1(x)}{g(x)}\right)^2\right) \\ g(x) = 1 + \dfrac{9}{n-1} \times \sum\limits_{i=2}^{n} x_i \\ x = (x_1, x_2, \ldots, x_n) \in [0.1]^n \end{cases}$	30	$x_i \in [0;1]$
T_5	$\begin{cases} minf_1(x) = x_1 \\ minf_2(x) = g(x) \times h(x) \\ g(x) = 1 + \dfrac{9}{n-1} \times \sum\limits_{i=2}^{n} x_i \\ h(x) = 1 - \sqrt{\dfrac{f_1(x)}{g(x)} - \dfrac{f_1(x)}{g(x)}} sin(10\pi f_1(x)) \\ x = (x_1, x_2, \ldots, x_n) \in [0.1]^n \end{cases}$	30	$x_i \in [0;1]$
T_6	$\begin{cases} minf_1(x) = x_1 \\ minf_2(x) = g(x) \times \sqrt{1 - \dfrac{f_1(x)}{g(x)}} \\ g(x) = 1 + \dfrac{9}{n-1} \times \sum\limits_{i=2}^{n} x_i \\ x = (x_1, x_2, \ldots, x_n) \in [0.1]^n \end{cases}$	30	$x_i \in [0;1]$

3.2.2. *Graphical results.* The graphical results are presented by the following graphs:

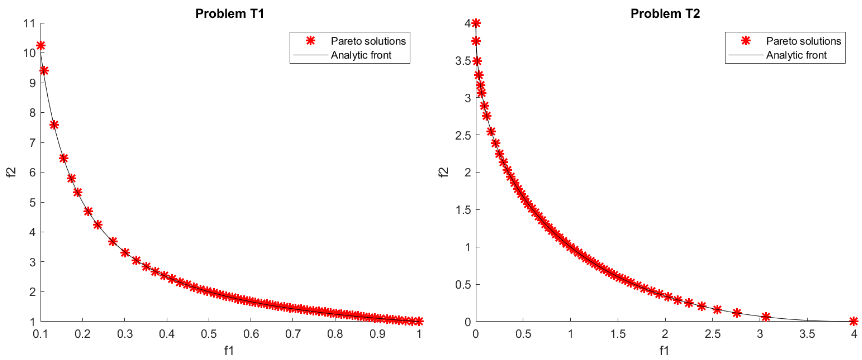

FIGURE 1. Pareto front of the problem T_1 et T_2.

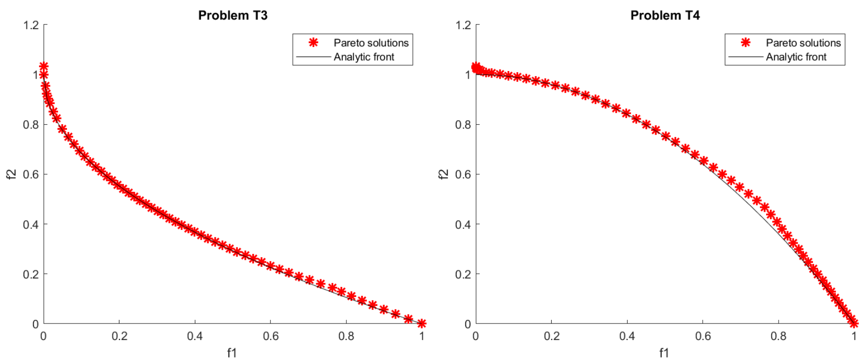

FIGURE 2. Pareto front of the problem T_3 et T_4.

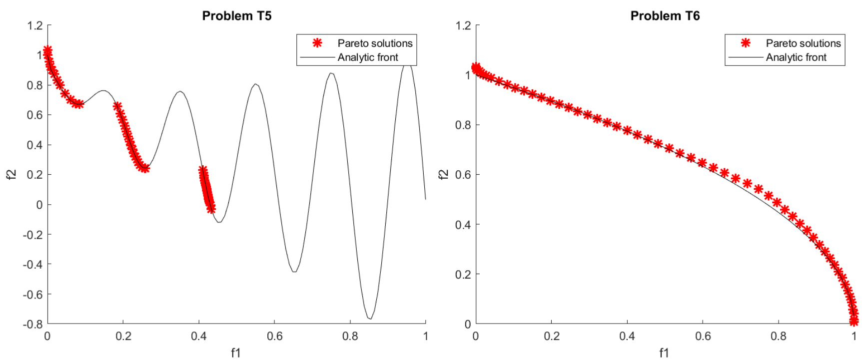

FIGURE 3. Pareto front of the problem T_5 et T_6.

3.2.3. *Numerical interpretations of simulation results.* The study of modified MOMA-Plus performances focuses on the convergence of obtained solutions to the analytical front and their distribution on the analytical front. When the values of these parameters are close to zero, a good performance is obtained. Performance indexes are defined by the relations below:

$$\Upsilon = \frac{\sqrt{\sum_{i=1}^{N} d_i^2}}{N} \quad \text{and} \quad \Lambda = \frac{d_f + d_l + \sum_{i=1}^{N-1} |d_i - \overline{d}|}{d_f + d_l + (N-1)\overline{d}}. \tag{3.7}$$

In these formulas, N denotes the number of solutions provided by our method. d_f and d_l define respectively the euclidean distances separating the upper and lower extremal solutions provided by our method. d_i is the euclidean distance between two consecutive solutions, $\overline{d}$ is the arithmetic average of all the solutions provided by our method. The performance index of the modified MOMA-plus method are shown in the table 2.

TABLE 2. Performance table

Modified MOMA-Plus	T_1	T_2	T_3	T_4	T_5	T_6
Υ	0.0077	0.0011	0.0044	0.0018	0.0029	0.0018
Λ	0.9819	0.9843	0.9823	0.9821	0.9823	0.9819

The coupling of MOMA-plus and exponential penalty function seems to converge quickly on all the multiobjective optimization problems studied. With regard to the values of the performance index, we have good convergence and good distribution of the Pareto optimal solutions.

Remember the value of the performance indices on convergences and distribution, provided by original MOMA-plus method [18]:

TABLE 3. Performance table of the MOMA-plus method

MOMA-plus	T_1	T_2	T_3	T_4	T_5	T_6
Υ	0.0691	0.0053	0.0046	0.0137	0.0599	0.0046
Λ	1.1833	0.5537	0.9820	0.3483	0.9835	0.9820

According to the performance index we can do the following comparisons on convergence indices :

TABLE 4. Comparison on convergence indices

Problems	T_1	T_2	T_3	T_4	T_5	T_6
modified MOMA-plus	**0.0077**	**0.0011**	**0.0044**	**0.0018**	**0.0029**	**0.0018**
MOMA-plus	0.0691	0.0053	0.0046	0.0137	0.0599	0.0046

On these six test problems, modified MOMA-Plus version is best than original version of MOMA-Plus. The distribution indices are defined in the table below:

TABLE 5. Comparison on distribution indices

Problems	T_1	T_2	T_3	T_4	T_5	T_6
modified MOMA-plus	**0.9819**	0.9843	0.9823	0.9821	**0.9823**	**0.9819**
MOMA-plus	1.1833	**0.5537**	**0.9820**	**0.3483**	0.9835	0.9820

On these six test problems, modified MOMA-Plus version is best than original version on tree problems.

4. CONCLUSION

Through this paper, we have demonstrated that it is possible to improve the performance of the MOMA-Plus method by proposing a new version. We have proved that by some theorems on the existence of Pareto optimal solutions using our method. In addition, we have confirmed that with numerical solutions on six test problems taken in the literature. This coupling of MOMA-Plus and the exponential penalty function has given solutions which are better than the original MOMA-plus method on the convergence criterion. But about how solutions are distributed on the Pareto front, the two methods are the same. So, we can conclude that the modified MOMA-Plus method is the best alternative for multiobjective optimization problems.

In the future, we intend to investigate the complexity of the MOMA-Plus method and its variants, as well as explore the possibility of tackling other types of optimization problems, such as combinational optimization and fuzzy optimization.

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