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THE BIVARIATE EXTENDED POISSON DISTRIBUTION OF TYPE 2

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ABSTRACT. In this paper we undertake the construction of a bivariate distribution generalising the univariate extended Poisson distribution by using the method of crossing laws, a method highlighted in [7]. We will call this law "the bivariate extended Poisson distribution of type 2", in reference to "the bivariate extended Poisson law of type 1" highlighted in [4]. We have shown that this law is a member of the family of bivariate Poisson distributions. Functional relations will be established between the two distributions.

1. INTRODUCTION

Several authors have studied bivariate Poisson distributions, in particular [2]. In [7], the authors evidenced the bivariate distribution of the univariate weighted Poisson distribution by the cross distribution method. [5] constructed the bivariate Com-Poisson law based on the law crossing method. Then in [3] the authors highlighted the weighted bivariate Poisson distribution with the bivariate Poisson distribution according to [2] as its base distribution; this distribution allows, by choosing the weight function, to generate all bivariate Poisson distributions. The bivariate Poisson distribution according to [2] is rightly considered as the standard distribution in $\mathbb{N}^2$ as is the Poisson distribution in $\mathbb{N}$. Finally, in [4] the

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authors constructed an extended bivariate Poisson distribution by the product of its marginal distributions by a factor; this distribution is called the "extended bivariate Poisson distribution of type 1". In this paper we will construct an extended bivariate Poisson distribution of type 2 obtained this time by the method of crossing distributions. We show that this law belongs to the family of bivariate Poisson laws and that the bivariate extended Poisson law of type 1 converges in law to the bivariate extended Poisson law of type 2. A simulation and applications will be carried out in another publication to show the interest of these statistical models.

2. A REVIEW OF DISTRIBUTIONS

2.1. The univariate weighted Poisson distribution. It is assumed that the realization y of a random variable Y with mass function $p(y;\theta)$ is attached with probability proportional to $w(y)$; the real y is the realization of a random variable Y^w called the weighted version of Y and which has mass function

$$P(Y^w = y) = \frac{w(y)}{E_\theta[w(Y)]}p(y;\theta), \quad y \in \mathbb{N},\ \theta \in \mathbb{R}_+^*, \tag{2.1}$$

$w(y)$ is called the weight function, a positive function and $E_\theta[w(Y)]$ the normalization constant such that $0 < E_\theta[w(Y)] < +\infty$. The distribution $p(y;\theta)$ is called the baseline distribution (cf. [3]). The function $w(y)$ may depend on a parameter ϕ, $w(y) = w(y,\phi)$ that represents the data recording mechanism. Note that $w(y)$ may also depend on the canonical parameter θ, $w(y) = w(y,\phi) = w(y,\phi,\theta)$. The law is said to be generated by the weight function. When the weight function $w(y) = w(y,\phi)$ does not depend on the canonical parameter, the weighted law has the following characteristics (cf. [9])

$$E_\theta(Y^w) = \theta\left(1 + \frac{d}{d\theta}\ln E_\theta[w(Y)]\right), \tag{2.2}$$

$$var(Y^w) = E_\theta(Y^w) + \theta^2\frac{d^2}{d\theta^2}\ln E_\theta[w(Y)]. \tag{2.3}$$

2.2. The univariate extended Poisson distribution.

Definition 2.1. *The univariate extended Poisson distribution has a mass function [6]*

$$P(Y=y)=\begin{cases}\dfrac{1-e^{-\theta}}{\beta} & y=0,\\ \left(\dfrac{\theta^y}{y!}e^{-\theta}\right)\beta^{-1}\left(\dfrac{\beta}{\theta}y-1\right) & y=1,2,\dots\end{cases}, \tag{2.4}$$

for all $\theta>0$ *and* $\beta\geq\theta$.

It can be written in the form [11]

$$P(Y=y)=\frac{\theta^y}{y!}e^{-\theta}\left(\frac{\beta}{\theta}y-1\right)\beta^{-1}\left\{\frac{1-e^{-\theta}}{\left(\dfrac{\beta}{\theta}y-1\right)e^{-\theta}}\right\}^{\delta_0(y)}, \tag{2.5}$$

for all $\theta>0$ and $\beta\geq\theta$, where δ_0 is the Dirac function in 0. θ is the canonic parameter. The extended Poisson distribution is a univariate weighted Poisson distribution with a weight function

$$w(y;\theta,\beta)=\left(\frac{\theta}{\beta}y-1\right)\left\{\frac{1-e^{-\theta}}{\left(\dfrac{\beta}{\theta}y-1\right)e^{-\theta}}\right\}^{\delta_0(y)}, \tag{2.6}$$

and normalization constant

$$E_\theta[w(Y;\theta,\beta)]=\beta. \tag{2.7}$$

The characteristics of this law cannot be calculated from expressions (2.2) and (2.3) because its weight function depends on the canonical parameter θ. This distribution has the following characteristics (cf [6, 11]):

$$E_\theta(Y)=1+\frac{\beta-1}{\beta}\theta, \tag{2.8}$$

$$var(Y)=\frac{\beta-1}{\beta^2}\theta^2+\frac{\beta+1}{\beta}\theta, \tag{2.9}$$

Under- or over-dispersed distribution. We have the following result

Proposition 2.1. *[4] The Fisher dispersion index of the variable* Y *which follows the extended Poisson distribution of parameters* (θ,β) *noted* $I(Y)$ *is such that:*

(i) $I(Y) > 1$ *if* $\dfrac{\beta}{1+\sqrt{\beta}} < \theta \leq \beta$, *i.e. the extended Poisson distribution is overdispersed ;*

(ii) $I(Y) < 1$ *if* $0 < \theta < \dfrac{\beta}{1+\sqrt{\beta}}$, *i.e. the extended Poisson distribution is underdispersed ;*

(iii) $I(Y) = 1$ *if* $\theta = \dfrac{\beta}{1+\sqrt{\beta}}$, *i.e. the extended Poisson distribution is equiderdispersed.*

Proposition 2.2. *[6] The generating function of the moments of the extended Poisson distribution is equal to*

$$M_Y(\eta) = \frac{1-(1-\beta e^{\eta})\, e^{\theta\,(e^{\eta}-1)}}{\beta}, \quad \eta \in [-1,1]. \tag{2.10}$$

2.3. **The bivariate Poisson distribution according to [2].**

Definition 2.2. *Let* Y_j $(j = 1,2)$ *be a random variable that follows the Poisson distribution of parameter* θ_j $(j = 1,2)$. *The vector* (Y_1, Y_2) *follows the bivariate Poisson distribution according to [2] if its mass function* f_{BP} *is equal to*

$$f_{BP}(y_1, y_2; \theta_1, \theta_2) = \left(\frac{\theta_1^{y_1}}{y_1!} e^{-\theta_1}\right) \left(\frac{\theta_2^{y_2}}{y_2!} e^{-\theta_2}\right), \quad (y_1, y_2) \in \mathbb{N}^2,\ (\theta_1, \theta_2) \in \mathbb{R}_+^{*2}, \tag{2.11}$$

under conditions

$$\log \theta_1 = x' \rho_1, \tag{2.12}$$

$$\log \theta_2 = x' \rho_2 + \eta y_1, \tag{2.13}$$

where $\rho_1,\ \rho_2$ *and* η *are parameters,* $x = (x_1, \ldots, x_p)$ *is a vector of deterministic variables or factors.*

The generalized model (2.12) has the response variable Y_1 and the model (2.13) the variable Y_2. The expression (2.12) induces that $P(Y_1 = y_1; \theta_1) = \left(\frac{\theta_1^{y_1}}{y_1!} e^{-\theta_1}\right)$ is a marginal law while the model (2.13) induces that $P(Y_2 = y_2; \theta_2) = P(Y_2 = y_2 / Y_1 = y_1) = \left(\frac{\theta_2^{y_2}}{y_2!} e^{-\theta_2}\right)$ is a conditional law. When $\eta = 0$ then the variables Y_1 and Y_2 are independent (see [4]).

The bivariate Poisson distribution according to [2] has the characteristics (see [1]).

$$E_{\theta_!}(Y_1) = var(Y_1) = \theta_1, \tag{2.14}$$

$$E_{\theta_2}(Y_2) = e^{x'\rho_2 + a_2 + \theta_1(e^{\eta} - 1)}, \tag{2.15}$$

$$var(Y_2) = E_{\theta_2}(Y_2)\,[E_{\theta_2}(Y_2)]^2 \left(e^{\theta_1(e^{\eta} - 1)} - 1\right), \tag{2.16}$$

$$cov(Y_1, Y_2) = \theta_1 E_{\theta_2}(Y_2)\left(e^{\eta} - 1\right). \tag{2.17}$$

Expression (2.16) shows that the variable Y_2 is overdispersed. From expression (2.17) the covariance is negative, null or positive depending on whether the parameter η is negative, null or positive.

2.4. **The bivariate extended Poisson distribution of type 1.** In [4] the authors evidenced the bivariate extended Poisson distribution obtained by the method of the product of its marginal distributions by a factor whose mass function noted $f_{BEP,1}$ by [4] is defined as follows

$$\begin{aligned} &f_{BEP,1}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2, \alpha) \\ &= \left(\prod_{j=1}^{2}\left[\left(\frac{\theta_j^{y_j}}{y_j!}e^{-\theta_j}\right)\left(\frac{\beta_j}{\theta_j}y_j - 1\right)\beta_j^{-1}\left\{\frac{1 - e^{-\theta_j}}{\left(\frac{\beta_j}{\theta_j}y_j - 1\right)e^{-\theta_j}}\right\}^{\delta_0(y_j)}\right]\right) \\ &\times g(y_1, y_2; \theta_1, \theta_2, \alpha), \end{aligned} \tag{2.18}$$

where $g(y_1, y_2; \theta_1, \theta_2, \alpha) = \left[1 + \alpha\left(e^{-y_1} - c_1\right)\left(e^{-y_2} - c_2\right)\right]$, with $c_j = E_{\theta_j}\left(e^{-Y_j}\right)$, $y_j \in \mathbb{N},\ \theta_j \in \mathbb{R}_+^*,\ \beta_j \geq \theta_j,\ (j = 1, 2)$ and $\alpha \in \mathbb{R}$.

We have the following result [4].

Proposition 2.3. *Under expressions* (2.12) *and* (2.13) *we have*

$$
\begin{aligned}
(2.19)\qquad & f_{BEP,1}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2, \alpha) \\
&= \prod_{j=1}^{2} \left[\left(\frac{\beta_j}{\theta_j} y_j - 1 \right) \beta_j^{-1} \left\{ \frac{1 - e^{-\theta_j}}{\left(\frac{\beta_j}{\theta_j} y_j - 1 \right) e^{-\theta_j}} \right\}^{\delta_0(y_j)} \right] \\
&\times g(y_1, y_2; \theta_1, \theta_2, \alpha) \times f_{BP}(y_1, y_2, \theta_1, \theta_2).
\end{aligned}
$$

Expression (2.19) confirms that the bivariate Poisson extended distribution of type 1 is a member of the family of bivariate Poisson distributions (see [3]).

Proposition 2.4. *[4]*

(i) *The marginal distributions of Y_1 and Y_2 are extended Poisson distributions of respective parameters (θ_1, β_1) and (θ_2, β_2).*

(ii)

$$
cov(Y_1, Y_2) = \alpha cov \left(Y_1, e^{-Y_1} \right) cov \left(Y_2, e^{-Y_2} \right). \tag{2.20}
$$

Corollary 2.1. *When $\alpha = 0$, the variables Y_1 and Y_2 are independent.*

3. The bivariate extended Poisson distribution of type 2

3.1. **Definition and first properties.** Since the univariate extended Poisson distribution is a univariate weighted Poisson distribution, its bivariate distribution of type 2 can be conceived as a bivariate weighted Poisson distribution as follows [7].

Definition 3.1. *Consider two variables Y_j $(j = 1, 2)$ of extended Poisson with parameters (θ_j, β_j) $j = 1, 2$. The bivariate distribution of the extended Poisson distribution according to the crossover method has the mass function $f_{BPE,2}$*

$$
\begin{aligned}
& f_{BEP,2}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2) \\
(3.1)\qquad &= \prod_{j=1}^{2} \left[\left(\frac{\theta_j^{y_j}}{y_j!} e^{-\theta_j} \right) \left(\frac{\beta_j}{\theta_j} y_j - 1 \right) \beta_j^{-1} \left\{ \frac{1 - e^{-\theta_j}}{\left(\frac{\beta_j}{\theta_j} y_j - 1 \right) e^{-\theta_j}} \right\}^{\delta_0(y_j)} \right],
\end{aligned}
$$

with $y_j \in \mathbb{N},\ \theta_j \in \mathbb{R}_+^*,$ *and* $\beta_j \geq \theta_j$ $(j = 1, 2)$*, under conditions* (2.12) *and* (2.13).

$f_{BEP,2}$ stands for "bivariate extended Poisson distribution of type 2". Referring to Section 2.3, $P(Y_1 = y_1)$ is the marginal distribution and $P(Y_2 = y_2) = P(Y_2 = y_2/Y_1 = y_1)$ the conditional. This leads to the following result.

Proposition 3.1. *If* $\eta = 0$*, then the variables* Y_1 *and* Y_2 *are independent.*

We have the following result

Proposition 3.2.

$$f_{BEP,2}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2) = \prod_{j=1}^{2} \left[\left(\frac{\beta_j}{\theta_j} y_j - 1 \right) \beta_j^{-1} \left\{ \frac{1 - e^{-\theta_j}}{\left(\frac{\beta_j}{\theta_j} y_j - 1 \right) e^{-\theta_j}} \right\}^{\delta_0(y_j)} \right] \times f_{BP}(y_1, y_2; \theta_1, \theta_2). \tag{3.2}$$

Proof. We have

$$f_{BEP,2}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2) = \left(\frac{\theta_1^{y_1}}{y_1!} e^{-\theta_1} \right) \left(\frac{\theta_2^{y_2}}{y_2!} e^{-\theta_2} \right) \times \prod_{j=1}^{2} \left[\left(\frac{\beta_j}{\theta_j} y_j - 1 \right) \beta_j^{-1} \left\{ \frac{1 - e^{-\theta_j}}{\left(\frac{\beta_j}{\theta_j} y_j - 1 \right) e^{-\theta_j}} \right\}^{\delta_0(y_j)} \right],$$

And under conditions (2.12) and (2.13) we are assured of the answer. □

Expression (3.2) confirms that the bivariate extended Poisson distribution of type 2 is a member of the family of bivariate Poisson distributions (see [3]).

The bivariate extended Poisson distribution of type 2 has the following characteristics.

Proposition 3.3.

(i)

$$E_{\theta_1}(Y_1) = 1 + \frac{\beta_1 - 1}{\beta_1} \theta_1. \tag{3.3}$$

(ii)

$$var(Y_1) = \frac{\beta_1 - 1}{\beta_1^2}\theta_1^2 + \frac{\beta_1 + 1}{\beta_1}\theta_1. \tag{3.4}$$

(iii)

$$E_{\theta_2}(Y_2) = 1 + \frac{\beta_2 - 1}{\beta_2} e^{x'\rho_2} \times \frac{1 - (1 - \beta_1 e^{\eta})\, e^{\theta_1 (e^{\eta} - 1)}}{\beta_1}. \tag{3.5}$$

(iv)

$$\begin{aligned}(3.6)\quad & var(Y_2) \\ &= \frac{\beta_2 - 1}{\beta_2^2} e^{2x'\rho_2} \times \frac{1 - \left(1 - \beta_1 e^{2\eta}\right) e^{\theta_1 \left(e^{2\eta} - 1\right)}}{\beta_1} \\ &\quad + \frac{\beta_2 + 1}{\beta_2} e^{x'\rho_2} \times \frac{1 - (1 - \beta_1 e^{\eta})\, e^{\theta_1 (e^{\eta} - 1)}}{\beta_1} + \left(\frac{\beta_2 - 1}{\beta_2}\right)^2 e^{2x'\rho_2} \\ (3.7)\quad &\quad \times \left[\frac{1 - \left(1 - \beta_1 e^{2\eta}\right) e^{\theta_1 \left(e^{2\eta} - 1\right)}}{\beta_1} - \left(\frac{1 - (1 - \beta_1 e^{\eta})\, e^{\theta_1 (e^{\eta} - 1)}}{\beta_1}\right)^2\right].\end{aligned}$$

(v)

$$cov(Y_1, Y_2) = \frac{\beta_2 - 1}{\beta_2} e^{x'\rho_2} e^{\theta_1 (e^{\eta} - 1)} \theta_1 e^{\eta} (e^{\eta} - 1). \tag{3.8}$$

The covariance is negative, null or positive depending on whether the parameter η is negative, null or positive.

Proof of proposition 7. Expressions (3.3) and (3.4) are obvious because the distribution of the variable Y_1 is a marginal distribution. We can recall that the generating function of the moments of the univariate Poisson distribution noted $M_Y(\eta)$ is equal to (cf. proposition 2)

$$M_Y(\eta) = \frac{1 - (1 - \beta e^{\eta})\, e^{\theta (e^{\eta} - 1)}}{\beta}, \quad \eta \in [-1, 1].$$

Since

$$E_{\theta_2}(Y_2) = E_{\theta_1}(E_{\theta_2}(Y_2|Y_1)),$$

and

$$E_{\theta_2}(Y_2|Y_1) = 1 + \frac{\beta_2 - 1}{\beta_2}\theta_2 = 1 + \frac{\beta_2 - 1}{\beta_2} e^{x'\rho_2 + \eta Y_1}.$$

Then

$$\begin{aligned} E_{\theta_2}(Y_2) =& E_{\theta_2}\left[1 + \frac{\beta_2 - 1}{\beta_2} e^{x'\rho_2 + \eta Y_1}\right] = 1 + \frac{\beta_2 - 1}{\beta_2} e^{x'\rho_2} E_{\theta_2}\left(e^{\eta Y_1}\right) \\ =& 1 + \frac{\beta_2 - 1}{\beta_2} e^{x'\rho_2} \times \frac{1 - (1 - \beta_1 e^{\eta})\, e^{\theta_1 (e^{\eta} - 1)}}{\beta_1}. \end{aligned}$$

Expression (3.5) is thus demonstrated.

We know that

$$var(Y_2) = E_{\theta_1}[var(Y_2|Y_1)] + var[E_{\theta_2}(Y_2|Y_1)].$$

The calculation gives us, firstly,

$$\begin{aligned} var(Y_2|Y_1) =& \frac{\beta_2 - 1}{\beta_2^2}\theta_2^2 + \frac{\beta_2 + 1}{\beta_2}\theta_2, \\ =& \frac{\beta_2 - 1}{\beta_2^2} e^{2x'\rho_2 + 2\eta Y_1} + \frac{\beta_2 + 1}{\beta_2} e^{x'\rho_2 + \eta Y_1}. \end{aligned}$$

Next,

$$\begin{aligned} E_{\theta_1}[var(Y_2|Y_1)] =& \frac{\beta_2 - 1}{\beta_2^2} e^{2x'\rho_2} e^{2\eta Y_1} + \frac{\beta_2 + 1}{\beta_2} e^{x'\rho_2} e^{\eta Y_1}, \\ =& \frac{\beta_2 - 1}{\beta_2^2} e^{2x'\rho_2} \times \frac{1 - \left(1 - \beta_1 e^{2\eta}\right) e^{\theta_1 (e^{2\eta} - 1)}}{\beta_1} \\ & + \frac{\beta_2 + 1}{\beta_2} e^{x'\rho_2} \times \frac{1 - (1 - \beta_1 e^{\eta})\, e^{\theta_1 (e^{\eta} - 1)}}{\beta_1}. \end{aligned}$$

Lastly,

$$\begin{aligned} var[E_{\theta_2}(Y_2|Y_1)] &= var\left(1 + \frac{\beta_2 - 1}{\beta_2} e^{x'\rho_2 + \eta Y_2}\right) \\ &= \left(\frac{\beta_2 - 1}{\beta_2}\right)^2 e^{2x'\rho_2} \times var\left(e^{\eta Y_1}\right) \\ &= \left(\frac{\beta_2 - 1}{\beta_2}\right)^2 e^{2x'\rho_2} \times \left[E_{\theta_2}\left(e^{2\eta Y_1}\right) - \left[E_{\theta_2}\left(e^{\eta Y_1}\right)\right]^2\right], \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\beta_2 - 1}{\beta_2}\right)^2 e^{2x'\rho_2} \\
&\quad \times \left[\frac{1-\left(1-\beta_1 e^{2\eta}\right) e^{\theta_1\left(e^{2\eta}-1\right)}}{\beta_1} - \left(\frac{1-\left(1-\beta_1 e^{\eta}\right) e^{\theta_1\left(e^{\eta}-1\right)}}{\beta_1}\right)^2\right].
\end{aligned}
$$

By summing the expressions $E_{\theta_1}[var(Y_2|Y_1)]$ and $var[E_{\theta_2}(Y_2|Y_1)]$, we find the expression (3.6).

We too know that

$$
cov(Y_1, Y_2) = E_{\theta_1,\theta_2}(Y_1Y_2) - E_{\theta_1}(Y_1)E_{\theta_2}(Y_2).
$$

We have,

$$
\begin{aligned}
E_{\theta_1,\theta_2}(Y_1Y_2) =& E_{\theta_1}[E_{\theta_2}(Y_1Y_2|Y_1)], \\
=& E_{\theta_1}[Y_1E_{\theta_2}(Y_2|Y_1)], \\
=& E_{\theta_1}\left[Y_1\left(1+\frac{\beta_2-1}{\beta_2}e^{x'\rho_2 + \eta Y_1}\right)\right], \\
=& E_{\theta_1}(Y_1) + \frac{\beta_2-1}{\beta_2}e^{x'\rho_2}E_{\theta_1}\left(Y_1e^{\eta Y_1}\right), \\
=& E_{\theta_1}(Y_1) + \frac{\beta_2-1}{\beta_2}e^{x'\rho_2}\frac{d}{d\eta}E_{\theta_1}\left(e^{\eta Y_1}\right), \\
=& 1 + \frac{\beta_1-1}{\beta_1}\theta + \frac{\beta_2-1}{\beta_2}e^{x'\rho_2} \times \frac{e^{\eta}e^{\theta_1\left(e^{\eta}-1\right)}\left[\beta_1 - \theta_1\left(1-\beta_1e^{\eta}\right)\right]}{\beta_1}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
cov(Y_1, Y_2) =& 1 + \frac{\beta_1-1}{\beta_1}\theta + \frac{\beta_2-1}{\beta_2}e^{x'\rho_2} \times \frac{e^{\eta}e^{\theta_1\left(e^{\eta}-1\right)}\left[\beta_1 - \theta_1\left(1-\beta_1e^{\eta}\right)\right]}{\beta_1} \\
&- \left(1+\frac{\beta_1-1}{\beta_1}\theta_1\right)\left(1+\frac{\beta_2-1}{\beta_2}e^{x'\rho_2} \times \frac{1-\left(1-\beta_1e^{\eta}\right)e^{\theta_1\left(e^{\eta}-1\right)}}{\beta_1}\right), \\
=& \frac{\beta_2-1}{\beta_1\beta_2}e^{x'\rho_2}e^{\theta_1\left(e^{\eta}-1\right)}\left[e^{\eta}[\beta_1 - \theta_1(1-\beta_1e^{\eta})]\right. \\
&\left. - \left(1+\frac{\beta_1-1}{\beta_1}\theta_1\right)[1-(1-\beta_1e^{\eta})]\right].
\end{aligned}
$$

Let's put $A = e^{\eta}[\beta_1 - \theta_1(1 - \beta_1 e^{\eta})]$ and $B = \left(1 + \frac{\beta_1 - 1}{\beta_1}\theta_1\right)[1 - (1 - \beta_1 e^{\eta})]$.

We have

$$cov(Y_1, Y_2) = \frac{\beta_2 - 1}{\beta_1 \beta_2} e^{x' \rho_2} e^{\theta_1 (e^{\eta} - 1)}(A - B).$$

By expanding the expression A, we find

$$A = \beta_1 e^{\eta} - \theta_1 e^{\eta} + \beta_1 \theta_1 e^{2\eta}.$$

If we also expand expression B, we find

$$B = \beta_1 e^{\eta} + (\beta_1 - 1)\theta_1 e^{2\eta},$$

and

$$A - B = \beta_1 \theta_1 \left(e^{2\eta} - e^{\eta}\right) = \beta_1 \theta_1 e^{\eta} \left(e^{\eta} - 1\right).$$

Then

$$cov(Y_1, Y_2) = \frac{\beta_2 - 1}{\beta_2} e^{x' \rho_2} e^{\theta_1 (e^{\eta} - 1)} \theta_1 e^{\eta} \left(e^{\eta} - 1\right).$$

And we are sure of the answer. □

3.2. **Functional relationships between the models $f_{BEP,1}$ and $f_{BEP,2}$.** Consider a sequence of real α_n $(n \in \mathbb{N})$ such that $\lim_{n\to\infty} \alpha_n = 0$. We can construct a family of bivariate extended Poisson distributions of type 1 $\{f_{BEP,n}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2),\ n \in \mathbb{N}\}$ such as $f_{BEP,n}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2) = f_{BEP,1}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2, \alpha_n)$.

We have the following result

Proposition 3.4. *The family of bivariate extended Poisson distributions of type 1 converges in distribution to the bivariate extended Poisson distribution of type 2.*

Proof. Indeed, we have

$$\lim_{n\to\infty} f_{BEP,n}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2) = f_{BEP,2}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2).$$

□

3.3. **Estimation of parameters.** These parameters will be estimated by the maximum likelihood method. Consider an $n-$sample $(y_{1,1}, y_{2,1}), (y_{1,2}, y_{2,2}), \ldots, (y_{1,n}, y_{2,n})$ of the pair of random variables (Y_1, Y_2) of density $f_{BEP,2}(y_1, y_2; \theta_1, \theta_2, \beta_1, \beta_2)$. Under conditions (2.12) and (2.13), substitution θ_1 and θ_2 by $\exp(x'\rho_1)$

and $\exp(x'\rho_2 + \eta y_1)$ respectively, the log-likelihood function $\log L = \log L(\rho_1, \rho_2, \beta_1, \beta_2, \eta; y_1, y_2; x)$ of $f_{BEP,3}$ is equal to

$$\begin{aligned}\log L = \sum_{i=1}^{n} \Bigg(\Big[& y_{1i} x_i' \rho_1 + (\delta_0(y_{1i}) - 1)\, e^{x_i'\rho_1} + \delta_0(y_{1i}) \log\left(1 - e^{-\exp(x_i'\rho_1)}\right) \\ & + (1 - \delta_0(y_{1i})) \log\left(\beta_1 e^{-x_i'\rho_1} y_{1i} - 1\right)\Big] \\ & + \Big[y_{2i}(x_i'\rho_2 + \eta y_{1i}) + (\delta_0(y_{2i}) - 1)\, e^{x_i'\rho_2 + \eta y_{1i}} \\ & + \delta_0(y_{2i}) \log\left(1 - e^{-\exp(x_i'\rho_2 + \eta y_{1i})}\right) \\ & + (1 - \delta_0(y_{2i})) \log\left(\beta_2 e^{-(x_i'\rho_2 + \eta y_{1i})} y_{2i} - 1\right)\Big]\Bigg) - n \sum_{j=1}^{2} \left[\log \beta_j + \overline{\log(y_j!)}\right],\end{aligned} \tag{3.9}$$

where $\overline{\log(y_j!)} = \dfrac{1}{n} \sum_{i=1}^{n} \log(y_{ji}!)$.

To evaluate the gradient vector, the first partial derivatives are defined

$$\begin{aligned}\frac{\partial}{\partial \rho_1} \ln L = \sum_{i=1}^{n} \Big[& y_{1i} x_i' - (\delta_0(y_{1i}) - 1)\, x_i'\theta_1 \\ & + \delta_0(y_{1i}) \frac{x_i'\theta_1 e^{\theta_1}}{1 - e^{-\theta_1}} + (1 - \delta_0(y_{1i})) \frac{-\beta_1 y_{1i} x_i' \theta_1}{\beta_1 y_{1i}\theta_1 - 1}\Big], \\ \frac{\partial}{\partial \rho_2} \ln L = \sum_{i=1}^{n} \Big[& y_{2i} x_i' - (\delta_0(y_{2i}) - 1)\, x_i'\theta_2 \\ & + \delta_0(y_{2i}) \frac{x_i'\theta_2 e^{\theta_2}}{1 - e^{-\theta_2}} + (1 - \delta_0(y_{2i})) \frac{-\beta_2 y_{2i} x_i' \theta_2}{\beta_2 y_{2i}\theta_2 - 1}\Big],\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial \beta_1} \ln L &= \sum_{i=1}^{n} (1 - \delta_0(y_{1i})) \frac{y_{1i}\theta_1^{-1}}{\beta_1 y_{1i}\theta_1^{-1} - 1} - \frac{n}{\beta_1}, \\ \frac{\partial}{\partial \beta_2} \ln L &= \sum_{i=1}^{n} (1 - \delta_0(y_{2i})) \frac{y_{2i}\theta_2^{-1}}{\beta_2 y_{2i}\theta_2^{-1} - 1} - \frac{n}{\beta_2}, \\ \frac{\partial}{\partial \eta} \ln L &= \sum_{i=1}^{n} \Big[y_{1i} y_{2i} - (\delta_0(y_{2i}) - 1)\, y_{1i\theta_2} \\ & \quad + \delta_0(y_{2i}) \frac{y_{1i}\theta_2 e^{-\theta_2}}{1 - e^{-\theta_2}} - (1 - \delta_0(y_{2i})) \frac{\beta_2 y_{1i} y_{2i} \theta_2^{-1}}{\beta_2 y_{2i}\theta_2^{-1} - 1}\Big].\end{aligned}$$

For the Hessian matrix, we do not present its analytical expression. In practice, it would be better to use the numerical hessian. The estimators $\widehat{\rho}_j,\ \widehat{\beta}_j,\ j = 1,2$ and $\widehat{\eta}$ can be obtained by using a calculation software such as the package maxLik for the statistical environment R (see [8]).

4. Conclusion

In this paper we have constructed the bivariate extended Poisson distribution of type 2 which generalises the univariate extended Poisson distribution. This distribution is conceived as a bivariate weighted Poisson distribution that results from a crossing of distributions. We have highlighted the conditions for the marginal distributions to be independent. We have also shown that this distribution is an element of the family of bivariate Poisson distributions. By comparing this distribution with the extended bivariate Poisson distribution of type 1 highlighted in [4], we have shown that the extended bivariate Poisson distribution of type 1 converges in distribution to the extended bivariate Poisson distribution of type 2. Simulations will be carried out in a forthcoming publication in order to make a practical study of these models.

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