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EULER MARUYAMA APPROXIMATIONS FOR A GENERAL CLASS OF STOCHASTIC FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study a class of stochastic fractional integro differential equations under the non-Lipschitz conditions. Thanks to Euler Maruyama's numerical scheme, we prove existence and uniqueness of a solution. We obtain also the strong convergence.

1. INTRODUCTION

An integrodifferential equation is an equation that involves both integrals and derivatives of an unknown function. The theory and applications of these equations play a very interesting role in the mathematical modeling of many fields taking into account the effects of real-world problems such as physics, biology, engineering sciences see among other ([3, 11, 19]).

Furthermore, over the past twenty years, with the advent of fractional differential equations as a field of research, more and more researchers have become interested in the study of fractional calculus. Since then, different concepts of derivatives and fractional integrals have been introduced, such as the integrals and derivatives of Riemann-Liouville, Caputo, Grunwald Letnikov, Riesz

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([8, 20]). Fractional differential equations ([17, 18]) can also effectively describe the dynamic behaviour of real-life phenomena with more accuracy than integer order equations.

With the development of fractional calculus, fractional integrodifferential equations appear in many domains such as electromagnetic waves ([29]) and dynamic population system ([33], [21]). Some authors ([1, 23]) showed local and global existence results of fractional integrodifferential equations.

On the other hand, the present of random effects is noted in many branches of science, in particular economics and finance, physics, demography, biology and medicine.

In 1942, Itô [13] introduced the stochastic differential equations (SDE) which will know a real expansion ([22, 28]).

Therefore stochastic integrodifferential equations appear as a natural extension of stochastic differential equations. These equations are involved in stochastic feedback systems ([27]), option pricing ([6]) and population growth model ([16, 33]).

Nowadays, more and more researchers are interested in fractional stochastic equations ([13, 14, 16]). Using the classical Picard-Lindelöf successive approximation scheme, Pedjeu and Ladde [26] established existence and uniqueness of solution of stochastic fractional differential equations. They used the Riemann-Liouville type fractional integral. The stochastic fractional equations are effective in modeling hereditary and hidden properties of certain noise systems in mathematical finance ([30]), in ecology and epidemiology ([26]). Also, note that several dynamic processes in science and engineering are under the influence of random perturbations of both internal and external environmental natures. So to bring more precision to the model, Pedjeu and Ladde [26] consider fractional stochastic differential equations by introducing the concept of dynamic processes operating under a set of linearly independent time scales.

In the literature we find a lot of result dealing with existence and uniqueness of fractional integrodifferential equations ([3–5, 15]). In this spirit Umamaheswari et al. [32] established an existence and uniqueness of solution of stochastic fractional integrodifferential equations [32] under the Lipschitz condition by using the Picard-Lindelöf approximation. These results generalise their initial work in [31] but also the results of Pedjeu and Ladde [26] by going from a stochastic fractional differential equation to a stochastic fractional integrodifferential equation.

However, for SDE with irregular coefficients (when the Lipschitz condition does not hold), Euler Maruyama's method plays a big role. This is why some authors have used this approach to justify the existence and uniqueness of solution of certain SDE. This is the case of Abramov et al. [2] when studying Constant Elasticity of Variance models with a non-lipschitz diffusion coefficient. In the same idea Hoang-Long and Taguchi [25] give the convergence rate of One-dimensional SDE with Holderian continuity drift coefficient. Xiaojie and Huijie [9], studying the SDE of McKean-Vlasov types, use Euler maruyama approximation and obtain a result of existence and unicity. Also Xinjie et al. [7] manages to show existence and uniqueness of nonlinear SDE by first establishing their connections with Voltera-type SDE. Nasroallah and Ouknine [24] study succesfully SDE with jump using an Euler Maruyama scheme and establish a mean square convergence.

The motivation of this paper is to weaken these conditions by considering an SDE for non Lipschitz coefficients. We use Euler Maruyama's scheme following the approach used by Xinjie et al. [7] to establish a result of existence and uniqueness of a general class of stochastic fractional integro-differential equations and prove also the strong convergence of order $\gamma \in (0, 1/2]$.

Our results will extend those of Umamaheswari et al. [32] to non-Lipschitz case and also the results of Xinjie et al. [7] to a general class of stochastic fractional integro-differential equations.

The rest of the paper is organized as follows. In section 1 we give some useful preliminaries results and the main hypotheses in this paper. We establish our main result in section 2. On one hand, our technical reside on the first version of Euler Maruyama scheme and on the other hand we use a modified Euler Maruyama scheme.

2. Preliminary

In this section we give some notions which will be used to prove our results.

Definition 2.1. $(Riemann - Liouville fractional\ integral)$. *Riemann-Liouville fractional integral operator of order* α *of a function* $f \in L^1(\mathbb{R}_+)$ *is defined by*

$$I_{0^+}^{(\alpha)}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \qquad \forall\alpha > 0,$$

where $\Gamma(.)$ *is the Euler Gamma function defined by*

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}\exp(-t)dt.$$

Definition 2.2. $(Riemann - Liouville\ fractional\ derivative)$. *The Riemann- Liouville fractional derivative of order* $\alpha > 0$*,* $n-1 < \alpha < n$*,* $n \in \mathbb{N}^*$*, is defined as*

$$D_{0^+}^{(\alpha)}f(t) = \left(\frac{d}{dt}\right)^n I_{0^+}^{(n-\alpha)}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}f(s)ds,$$

where the function $f(t)$ *has absolutely continuous derivatives up to order* $(n-1)$.

Definition 2.3. *(Multi-time scale Integral [26]) For* $p \in \mathbb{N}$*,* $p > 1$*, let* $\{T_1, \cdots, T_p\}$ *be a set of linearly independent time scales. Let* $f : [a, b[\times\mathbb{R}^{p-1} \to \mathbb{R}^n$ *be a continuous function defined by* $f(t) = f(T_1(t), \cdots, T_p(t))$*. The multi-time scale integral of the composite function* f *over an interval* $[t_0, t] \subseteq]a, b[$ *is defined as the sum of* p *integrals with respect to the time-scales* $T_1, \cdots, T_p$*. We denote it by* If*,*

$$If(t) := \sum_{j=1}^{p} I_j f(t),$$

where the sense of the integral

$$I_j f(t) = \int_{t_0}^t f(s)dT_j(s)$$

depends on the time scale T_j*, for each* $j = 1, \cdots, p$.

Example 1. *For* $p = 3$*, consider the linearly independent set consisting of time scales* $T_1(t) := t$*,* $T_2(t) := B(t)$ *where* B *is the standard Wiener process, and* $T_3(t) := t^\alpha$*,* $0 < \alpha < 1$ *as defined before. In this case,*

$$f(t) \equiv f(T_1(t), T_2(t), T_3(t)) \quad and \quad If(t) = I_1 f(t) + I_2 f(t) + I_3 f(t),$$

where the integrals

$$I_1 f(t) = \int_{t_0}^t f(s)ds, \quad I_2 f(t) = \int_{t_0}^t f(s)dB(s), \quad I_3 f(t) = \int_{t_0}^t f(s)(ds)^\alpha$$

are Riemann, Itô -Doob, and Riemann–Liouville type, respectively.

Under the set of time scales in this example, we have the following stochastic fractional differential equation

$$dx(t) = b(t, x(t))dt + \sigma_1(t, x(t))dW_t + \sigma_2(t, x(t))(dt)^\alpha, \quad t \in [0, T]. \tag{2.1}$$

Remark 2.1.

(i) *If* $\sigma_2 = 0$*, then the SDE* (2.1) *reduced to known Itô-Doob type stochastic*

$$dx(t) = b(t, x(t))dt + \sigma_1(t, x(t))dW_t, \qquad x(0) = x_0$$

whose fundamental properties and applications have been well studied for more than half-century.

(ii) *If* $\sigma_1 = 0$*, we have the following generalized version of the classical deterministic fractional differential equations*

$$dx(t) = b(t, x(t))dt + \sigma_2(t, x(t))(dt)^\alpha, \qquad x(0) = x_0.$$

(iii) *If* $\sigma_1 = 0$ *and* $\sigma_2 = 0$*, then equation* (2.1) *is to the deterministic*

$$dx(t) = b(t, x(t))dt, \qquad x(0) = x_0.$$

The following lemma known (Bihari inequality (Mao [22, Theorem 8.2, Chapter 1])) will play a big role in this work.

Lemma 2.1. *(Bihari Inequality) Let* $T > 0$*,* $u_0 \geq 0$*,* $u(t), v(t)$ *be continuous nonnegative functions on* $[0, T]$*, and* $\rho : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *be a concave continuous nondecreasing function such that* $\rho(x) > 0$ *for* $x > 0$*. If*

$$u(t) \leq u_0 + \int_0^t v(s)\rho\left(u(s)\right)ds, \qquad \forall t \in [0, T],$$

then

$$u(t) \leq G^{-1}\left(G(u_0) + \int_0^t v(s)ds\right)$$

holds for all $t \in [0, T]$ *such that*

$$G(u_0) + \int_0^t v(s)ds \in Dom(G^{-1}),$$

where

$$G(x) = \int_1^x \frac{ds}{\rho(s)}, \qquad x \geq 0.$$

G^{-1} is the inverse function of G and $Dom(G^{-1})$ is the domain $G^{-1}(.)$. In particular, if $u_0 = 0$ and

$$\int_{0^+} \frac{ds}{\rho(s)} = +\infty,$$

then $u(t) = 0$ for all $t \in [0, T]$.

3. Hypotheses

Let Ω be a non-empty set, $\mathcal{F}$ a σ-algebra of sets of Ω and $\mathbb{P}$ a probability measure defined on $\mathcal{F}$. Assuming $W = (W_t)_{t\geq 0}$ be an m-dimensional Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a fix real $T > 0$, let $J = [0, T]$. Given a real $\alpha \in]1/2, 1[$ and X_0 a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of the σ-algebra $(\mathcal{F}_t)_{t\in J}$ generated by $\{W_s, 0 \leq s \leq t\}$. Define the filtration $\mathcal{F}_t^0$ generated by X_0 and $\{W_s, 0 \leq s \leq t\}$. Let $b, \sigma_2 \in \mathcal{C}\left(J \times \mathbb{R}^p \times \mathbb{R}^{p\times p}, \mathbb{R}^p\right)$, $\sigma_1 \in \mathcal{C}\left(J \times \mathbb{R}^p \times \mathbb{R}^{p\times p}, \mathbb{R}^{p\times m}\right)$ and $f_i, g_i, h_i \in \mathcal{C}(J \times J \times \mathbb{R}^p, \mathbb{R}^p)$, $i = 1, 2, \cdots, p$.

Consider the following stochastic fractional integrodifferential equation: for all $t \in J$,

$$\begin{aligned} & dX_t \\ = \ & b\left(t, X_t, \int_0^t f_1(t,s,X_s)ds, \int_0^t f_2(t,s,X_s)ds, \cdots, \int_0^t f_p(t,s,X_s)ds\right) dt \\ + \ & \sigma_1\left(t, X_t, \int_0^t g_1(t,s,X_s)ds, \int_0^t g_2(t,s,X_s)ds, \cdots, \int_0^t g_p(t,s,X_s)ds\right) dW_t \\ + \ & \sigma_2\left(t, X_t, \int_0^t h_1(t,s,X_s)ds, \int_0^t h_2(t,s,X_s)ds, \cdots, h_p(t,s,X_s)ds\right) (dt)^\alpha \end{aligned} \tag{3.1}$$

We make the following assumptions to study existence and uniqueness of solution to this equation. For $i = 1, 2, \ldots, p$, let,

$$F_i = \int_0^s f_i(s,u,x_u)du, \quad G_i = \int_0^s g_i(s,u,x_u)du, \quad H_i = \int_0^s h_i(s,u,x_u)du,$$

and

$$\widetilde{F}_i = \int_0^s f_i(s,u,\widetilde{x}_u)du, \quad \widetilde{G}_i = \int_0^s g_i(s,u,\widetilde{x}_u)du, \quad \widetilde{H}_i = \int_0^s h_i(s,u,\widetilde{x}_u)du.$$

Assumption 1 (A_1): Linear growth condition: Let $x \in \mathbb{R}^p$

$$
\begin{aligned}
(A_{1-2}): \quad & \mid b\,(t,x,F_1,F_2,\ldots,F_p) \mid^2 + \mid \sigma_1\,(t,x,G_1,G_2,\ldots,G_p) \mid^2 \\
& + \mid \sigma_2\,(t,x,H_1,H_2,\ldots,H_p) \mid^2 \\
& \leq K^2 \left(1+ \mid x \mid^2 + \sum_{i=1}^{p} \mid F_i \mid^2 + \sum_{i=1}^{p} \mid G_i \mid^2 + \sum_{i=1}^{p} \mid H_i \mid^2 \right), \\
(A_{1-2}): \quad & \mid F_i \mid^2 + \mid G_i \mid^2 + \mid H_i \mid^2 \leqslant k_i^2 \left(1+ \mid x \mid^2\right),
\end{aligned}
$$

where $K, k_i > 0$, i=1,2,..., p, are constants.

Assumption 2 (A_2): The non Lipschitz condition: Let $x, \widetilde{x} \in \mathbb{R}^p$:

$$
\begin{aligned}
(A_{2-1}): \quad & \mid b\,(t,x,F_1,F_2,\ldots,F_p) - b\left(t,\widetilde{x},\widetilde{F}_1,\widetilde{F}_2,\ldots,\widetilde{F}_p\right) \mid^2 \\
& \leq L\rho\left(\mid x-\widetilde{x} \mid^2\right) + \delta_1 \sum_{i=1}^{p} \mid F_i - \widetilde{F}_i \mid^2, \\
(A_{2-2}): \quad & \mid \sigma_1\,(t,x,G_1,G_2,\ldots,G_p) - \sigma_1\left(t,\widetilde{x},\widetilde{G}_1,\widetilde{G}_2,\ldots,\widetilde{G}_p\right) \mid^2 \\
& \leq L\rho\left(\mid x-\widetilde{x} \mid^2\right) + \delta_2 \sum_{i=1}^{p} \mid G_i - \widetilde{G}_i \mid^2, \\
(A_{2-3}): \quad & \mid \sigma_2\,(t,x,H_1,H_2,\ldots,H_p) - \sigma_2\left(t,\widetilde{x},\widetilde{H}_1,\widetilde{H}_2,\ldots,\widetilde{H}_p\right) \mid^2 \\
& \leq L\rho\left(\mid x-\widetilde{x} \mid^2\right) + \delta_3 \sum_{i=1}^{p} \mid H_i - \widetilde{H}_i \mid^2, \\
(A_{2-4}): \quad & \mid F_i - \widetilde{F}_i \mid^2 + \mid G_i - \widetilde{G}_i \mid^2 + \mid H_i - \widetilde{H}_i \mid^2 \leqslant L_i^2 \left(\mid x-\widetilde{x} \mid^2\right),
\end{aligned}
$$

where L, L_i, i=1,2,..., p and $\delta_j, j = 1, \ldots 3$ are positive constants. ρ is a concave non-decreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\rho(0) = 0$, $\rho(v) > 0$ for $v > 0$ and

$$
\int_{0^+} \frac{ds}{\rho(s)} = +\infty.
$$

Assumption 3 (A_3): There exist a positive constant $\widetilde{k}$ such that, for $x \in \mathbb{R}^p$ and $\forall t_1, t_2 \in [0,T]$:

$$\begin{aligned}
(A_{3-1}): \quad & \mid b\left(t_1, x, F_1, F_2, \ldots, F_p\right) - b\left(t_2, x, F_1, F_2, \ldots, F_p\right) \mid \\
& \leqslant \widetilde{k}\left(1+\mid x \mid\right) \mid t_1 - t_2 \mid \\
(A_{3-2}): \quad & \mid \sigma_1\left(t_1, x, G_1, G_2, \ldots, G_p\right) - \sigma_1\left(t_2, x, G_1, G_2, \ldots, G_p\right) \mid \\
& \leqslant \widetilde{k}\left(1+\mid x \mid\right) \mid t_1 - t_2 \mid, \\
(A_{3-3}): \quad & \mid \sigma_2\left(t_1, x, H_1, H_2, \ldots, H_p\right) - \sigma_2\left(t_2, x, H_1, H_2, \ldots, H_p\right) \mid \\
& \leqslant \widetilde{k}\left(1+\mid x \mid\right) \mid t_1 - t_2 \mid .
\end{aligned}$$

4. Existence and uniqueness

The initial equation (3.1) can be rewritten via the definition 2.1 into the following equivalent equation:

$$\begin{aligned}
X_t = X_0 \quad + \quad & \int_0^t b\left(s, X_s, \int_0^s f_1(s,u,X_u)du, \int_0^s f_2(s,u,X_u)du, \ldots,\right. \\
& \left.\int_0^s f_p(s,u,X_u)du\right) ds + \int_0^t \sigma_1\left(s, X_s, \int_0^s g_1(s,u,X_u)du,\right. \quad (4.1) \\
& \left.\int_0^s g_2(s,u,X_u)du, \ldots, \int_0^s g_p(s,u,X_u)du\right) dW(s) \quad (4.2) \\
+ \quad & \alpha \int_0^t (t_n - s)^{\alpha-1} \sigma_2\left(s, X_s, \int_0^s h_1(s,u,X_u)du,\right. \quad (4.3) \\
& \left.\int_0^s h_2(s,u,X_u)du, \ldots, \int_0^s h_p(s,u,X_u)du\right) ds.
\end{aligned}$$

In the following we will study equation (4.1) by using Euler Maruyama scheme.

4.1. Euler Maruyama scheme.

For every integer $N \geq 1$, we consider the following uniform subdivision of $J = [0,T]$:

$$\tau_N = \{t_n := n\tfrac{T}{N}, n = 0, 1, \ldots, N\}.$$

By taking $t = t_n$ in equation (4.1) we get

$$\begin{aligned} X_{t_n} = X_0 \quad + & \quad \int_0^{t_n} b\left(s, X_s, \int_0^s f_1(s,u,X_u)du, \int_0^s f_2(s,u,X_u)du, \dots, \right. \\ & \left. \int_0^s f_p(s,u,X_u)du\right) ds + \int_0^{t_n} \sigma_1\left(s, X_s, \int_0^s g_1(s,u,X_u)du, \right. \\ & \left. \int_0^s g_2(s,u,X_u)du, \dots, \int_0^s g_p(s,u,X_u)du\right) dW(s) \\ + & \quad \alpha \int_0^{t_n} (t_n - s)^{\alpha-1} \sigma_2\left(s, X_s, \int_0^s h_1(s,u,X_u)du, \right. \\ & \left. \int_0^s h_2(s,u,X_u)du, \dots, \int_0^s h_p(s,u,X_u)du\right) ds, \end{aligned}$$

and using the subdivision we obtain

$$\begin{aligned} X_{t_n} = X_0 \quad + & \quad \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} b\left(s, X_s, \int_0^s f_1(s,u,X_u)du, \int_0^s f_2(s,u,X_u)du, \dots, \right. \\ & \left. \int_0^s f_p(s,u,X_u)du\right) ds + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sigma_1\left(s, X_s, \int_0^s g_1(s,u,X_u)du, \right. \\ & \left. \int_0^s g_2(s,u,X_u)du, \dots, \int_0^s g_p(s,u,X_u)du\right) dW(s) \\ + & \quad \alpha \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_n - s)^{\alpha-1} \sigma_2\left(s, X_s, \int_0^s h_1(s,u,X_u)du, \right. \\ & \left. \int_0^s h_2(s,u,X_u)du, \dots, \int_0^s h_p(s,u,X_u)du\right) ds. \end{aligned}$$

For $s \in [t_i, t_{i+1}[$ we can approximate $X_s \approx X_{t_i}$ and then we have

$$\begin{aligned} X_{t_n} \approx X_0 \quad + & \quad \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} b\left(s, X_{t_i}, \int_0^s f_1(s,u,X_u)du, \int_0^s f_2(s,u,X_u)du, \dots, \right. \\ & \left. \int_0^s f_p(s,u,X_u)du\right) ds + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sigma_1\left(s, X_{t_i}, \int_0^s g_1(s,u,X_u)du, \right. \\ & \left. \int_0^s g_2(s,u,X_u)du, \dots, \int_0^s g_p(s,u,X_u)du\right) dW(s) \end{aligned}$$

$$+ \quad \alpha \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_n - s)^{\alpha-1} \sigma_2 \left(s, X_{t_i}, \int_0^s h_1(s,u,X_u)du, \right.$$
$$\left. \int_0^s h_2(s,u,X_u)du, \ldots, \int_0^s h_p(s,u,X_u)du \right) ds.$$

Consider the new process defined by

$$\begin{aligned} X_{t_n}^N = X_0 \quad &+ \quad \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} b \left(s, X_{t_i}^N, \int_0^s f_1(s,u,X_u)du, \int_0^s f_2(s,u,X_u)du, \ldots, \right. \\ &\left. \int_0^s f_p(s,u,X_u)du \right) ds + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sigma_1 \left(s, X_{t_i}^N, \int_0^s g_1(s,u,X_u)du, \right. \\ &\left. \int_0^s g_2(s,u,X_u)du, \ldots, \int_0^s g_p(s,u,X_u)du \right) dW(s) \\ &+ \quad \alpha \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_n - s)^{\alpha-1} \sigma_2 \left(s, X_{t_i}^N, \int_0^s h_1(s,u,X_u)du, \right. \\ &\left. \int_0^s h_2(s,u,X_u)du, \ldots, \int_0^s h_p(s,u,X_u)du \right) ds. \end{aligned}$$

Let

$$\widehat{X}_t^N = \sum_{n=0}^{N} X_{t_n}^N \mathbb{1}_{[t_n, t_{n+1}[}(t), \qquad t \in [0,T],$$

the Euler Maruyama's scheme of equation (4.1) is defined as follows:

$$\begin{aligned} X_t^N = X_0 \quad &+ \quad \int_0^t b \left(s, \widehat{X}_s^N, \int_0^s f_1(s,u,\widehat{X}_u^N)du, \int_0^s f_2(s,u,\widehat{X}_u^N)du, \ldots, \right. \\ &\left. \int_0^s f_p(s,u,\widehat{X}_u^N)du \right) ds + \int_0^t \sigma_1 \left(s, \widehat{X}_s^N, \int_0^s g_1(s,u,\widehat{X}_u^N)du, \right. \\ &\left. \int_0^s g_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s g_p(s,u,\widehat{X}_u^N)du \right) dW(s) \\ &+ \quad \alpha \int_0^t (t-s)^{\alpha-1} \sigma_2 \left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du, \right. \\ &\left. \int_0^s h_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s h_p(s,u,\widehat{X}_u^N)du \right) ds. \end{aligned} \tag{4.4}$$

Theorem 4.1. *Under the Assumptions (A_1)and (A_2), there exist a unique solution of equation* (4.1). *Moreover, for all $t \in [0,T]$, there exist a positive constant C independent of N such that*

$$\mathbb{E} \mid X_t \mid^2 \leq C \qquad \textit{and} \qquad \lim_{N\to+\infty} \mathbb{E} \mid X_t^N - X_t \mid^2 = 0,$$

where $N \geq 1$ and X_t^N are defined by (4.4).

For the proof of this theorem we will need the following two results.

Lemma 4.1. *Under the Assumptions (A_1) and (A_2), there exist a positive constant C independent of N such that*

$$\mathbb{E} \mid \widehat{X}_t^N \mid^2 \leq C \qquad \textit{and} \qquad \mathbb{E} \mid X_t^N \mid^2 \leq C.$$

Proof. By Cauchy–Schwarz inequality, we obtain from equation (4.4) that

$$\begin{aligned} & \mid X_t^N \mid \\ \leq & \mid X_0 \mid + T^{1/2} \left(\int_0^t \left| b \left(s, \widehat{X}_s^N, \int_0^s f_1(s,u,\widehat{X}_u^N)du, \int_0^s f_2(s,u,\widehat{X}_u^N)du, \right.\right.\right. \\ & \left.\left.\left. \ldots, \int_0^s f_p(s,u,\widehat{X}_u^N)du \right) \right|^2 ds \right)^{1/2} + \left| \int_0^t \sigma_1 \left(s, \widehat{X}_s^N, \int_0^s g_1(s,u,\widehat{X}_u^N)du, \right.\right. \\ & \left.\left. \int_0^s g_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s g_p(s,u,\widehat{X}_u^N)du \right) dW(s) \right| \\ + & \left(\alpha^2 \int_0^t (t-s)^{2\alpha-2} \, ds \right)^{\frac{1}{2}} \left(\int_0^t \left| \sigma_2 \left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du, \int_0^s \right.\right.\right. \\ & \left.\left.\left. h_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s h_p(s,u,\widehat{X}_u^N)du \right) \right|^2 ds \right)^{\frac{1}{2}}, \end{aligned}$$

applying the algebraic inequality $(a+b+c+d)^2 \leq 4\left(a^2+b^2+c^2+d^2\right)$

$$\begin{aligned}
\mid X_t^N \mid^2 \ \leq \ & 4\Big(\mid X_0 \mid^2 + T\int_0^t \Big| b\left(s, \widehat{X}_s^N, \int_0^s f_1(s,u,\widehat{X}_u^N)du, \int_0^s f_2(s,u,\widehat{X}_u^N)du,\right.\\
& \left. \ldots, \int_0^s f_p(s,u,\widehat{X}_u^N)du\right) \Big|^2 ds + \left| \int_0^t \sigma_1\left(s, \widehat{X}_s^N, \int_0^s g_1(s,u,\widehat{X}_u^N)du,\right.\right.\\
& \left.\left. \int_0^s g_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s g_p(s,u,\widehat{X}_u^N)du\right) dW(s)\right|^2\\
+ \ & \alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \Big|\sigma_2\left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du,\right.\\
& \left. \int_0^s h_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s h_p(s,u,\widehat{X}_u^N)du\right) \Big|^2 ds\Big),
\end{aligned}$$

taking the mathematical expectation

$$\begin{aligned}
& \mathbb{E} \mid X_t^N \mid^2\\
\leq \ & 4\Big[\mid \mathbb{E}X_0 \mid^2 + T\int_0^t \mathbb{E}\Big| b\left(s, \widehat{X}_s^N, \int_0^s f_1(s,u,\widehat{X}_u^N)du, \int_0^s f_2(s,u,\widehat{X}_u^N)du,\right.\\
& \left. \ldots, \int_0^s f_p(s,u,\widehat{X}_u^N)du\right) \Big|^2 ds + \mathbb{E}\left| \int_0^t \sigma_1\left(s, \widehat{X}_s^N, \int_0^s g_1(s,u,\widehat{X}_u^N)du,\right.\right.\\
& \left.\left. \int_0^s g_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s g_p(s,u,\widehat{X}_u^N)du\right) dW(s)\right|^2\\
& + \alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\Big|\sigma_2\left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du, \int_0^s h_2(s,u,\widehat{X}_u^N)du,\right.\\
& \left. \ldots, \int_0^s h_p(s,u,\widehat{X}_u^N)du\right) \Big|^2 ds\Big],
\end{aligned}$$

the Itô isometry and the linear growth give

$$\begin{aligned}
\mathbb{E} \mid X_t^N \mid^2 \ \leq \ & 4\Big[\mathbb{E} \mid X_0 \mid^2 + T\int_0^t K^2\left(1 + \sum_{i=1}^p k_i^2\right)\left(1 + \mathbb{E} \mid \widehat{X}_s^N \mid^2\right) ds\\
+ \ & \int_0^t K^2\left(1 + \sum_{i=1}^p k_i^2\right)\left(1 + \mathbb{E} \mid \widehat{X}_s^N \mid^2\right) ds\\
+ \ & \alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t K^2\left(1 + \sum_{i=1}^p k_i^2\right)\left(1 + \mathbb{E} \mid \widehat{X}_s^N \mid^2\right) ds\Big]
\end{aligned}$$

$$\leq \quad 4\bigg[\mid \mathbb{E}X_0 \mid^2 + K^2 \left(T + 1 + \alpha^2 \frac{T^{2\alpha-1}}{2\alpha - 1}\right) \left(1 + \sum_{i=1}^{p} k_i^2\right) \int_0^t \left(1 + \mathbb{E} \mid \widehat{X}_s^N \mid^2\right) ds\bigg].$$

Let $C_1 = 4K^2 \left(T + 1 + \alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1}\right) (1 + \sum_{i=1}^{p} k_i^2)$, we have

$$\mathbb{E} \mid X_t^N \mid^2 \leq 4\mathbb{E} \mid X_0 \mid^2 + C_1 T + C_1 \int_0^t \mathbb{E} \mid \widehat{X}_s^N \mid^2 ds.$$

But there exist a unique integer n such that for $t \in [t_n, t_{n+1}[$, we have $X_{t_n}^N = \widehat{X}_t^N$. Then

$$\begin{aligned} \mathbb{E} \mid \widehat{X}_t^N \mid^2 = \mathbb{E} \mid X_{t_n}^N \mid^2 &\leq 4\mathbb{E} \mid X_0 \mid^2 + C_1 T + C_1 \int_0^{t_n} \mathbb{E} \mid \widehat{X}_s^N \mid^2 ds \\ &\leq 4\mathbb{E} \mid X_0 \mid^2 + C_1 T + C_1 \int_0^{t} \mathbb{E} \mid \widehat{X}_s^N \mid^2 ds. \end{aligned}$$

Applying the Gronwall inequality

$$\mathbb{E} \mid \widehat{X}_t^N \mid^2 \leq C,$$

where $C = 4\mathbb{E} \mid X_0 \mid^2 + C_1 T \exp(C_1 T)$, and by the continuity of X_t^N we obtain

$$\mathbb{E} \mid X_t^N \mid^2 \leq C.$$

□

Lemma 4.2. *Under the Assumptions (A_1) and (A_2), there exist a positive constant C independent of N and $0 < \gamma \leq 1$ such that*

$$\mathbb{E} \mid X_t^N - \widehat{X}_t^N \mid^2 \leq \frac{C}{N^\gamma}, \qquad \forall t \in [0, T].$$

Proof. For arbitrary $t \in [0, T]$, there is a unique integer n such that $t \in [t_n, t_{n+1}[$, and $\widehat{X}_t^N = X_{t_n}^N$. So it follows from equation (4.4) that

$$\begin{aligned} X_t^N - \widehat{X}_t^N &= X_t^N - X_{t_n}^N = \int_{t_n}^t b\left(s, \widehat{X}_s^N, \int_0^s f_1(s,u,\widehat{X}_u^N)du, \int_0^s f_2(s,u,\widehat{X}_u^N)du,\right. \\ & \left. \dots, \int_0^s f_p(s,u,\widehat{X}_u^N)du\right) ds + \int_{t_n}^t \sigma_1\left(s, \widehat{X}_s^N, \int_0^s g_1(s,u,\widehat{X}_u^N)du,\right. \\ & \left. \int_0^s g_2(s,u,\widehat{X}_u^N)du, \dots, \int_0^s g_p(s,u,\widehat{X}_u^N)du\right) dW(s) \\ &+ \alpha \int_0^{t_n} \left[(t-s)^{\alpha-1} - (t_n - s)^{\alpha-1}\right] \sigma_2\left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du, \int_0^s\right. \\ & \left. h_2(s,u,\widehat{X}_u^N)du, \dots, \int_0^s h_p(s,u,\widehat{X}_u^N)du\right) ds \\ &+ \alpha \int_{t_n}^t (t-s)^{\alpha-1} \sigma_2\left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du, \int_0^s h_2(s,u,\widehat{X}_u^N)du,\ t\right. \\ & \left. \dots, \int_0^s h_p(s,u,\widehat{X}_u^N)du\right) ds. \end{aligned}$$

The Cauchy-Schwarz inequality

$$\begin{aligned} & \mid X_t^N - \widehat{X}_t^N \mid \\ \leq\ & T^{1/2}\left(\int_{t_n}^t \left| b\left(s, \widehat{X}_s^N, \int_0^s f_1(s,u,\widehat{X}_u^N)du, \int_0^s f_2(s,u,\widehat{X}_u^N)du,\right.\right.\right. \\ & \left.\left.\left. \dots, \int_0^s f_p(s,u,\widehat{X}_u^N)du\right) \right|^2 ds\right)^{1/2} + \left| \int_{t_n}^t \sigma_1\left(s, \widehat{X}_s^N, \int_0^s g_1(s,u,\widehat{X}_u^N)du,\right.\right. \\ & \left.\left. \int_0^s g_2(s,u,\widehat{X}_u^N)du, \dots, \int_0^s g_p(s,u,\widehat{X}_u^N)du\right) dW(s)\right| \\ +\ & \alpha \int_0^{t_n} \left[(t-s)^{\alpha-1} - (t_n - s)^{\alpha-1}\right] \sigma_2\left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du,\right. \\ & \left. \int_0^s h_2(s,u,\widehat{X}_u^N)du, \dots, \int_0^s h_p(s,u,\widehat{X}_u^N)du\right) ds \\ +\ & \left(\alpha^2 \int_{t_n}^t (t-s)^{2\alpha-2} ds\right)^{\frac{1}{2}} \left(\int_{t_n}^t \left| \sigma_2\left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du, \int_0^s\right.\right.\right. \\ & \left.\left.\left. h_2(s,u,\widehat{X}_u^N)du, \dots, \int_0^s h_p(s,u,\widehat{X}_u^N)du\right) \right|^2 ds\right)^{\frac{1}{2}}, \end{aligned}$$

applying the algebraic inequality $(a+b+c+d)^2 \leq 4\left(a^2+b^2+c^2+d^2\right)$, we have

$$\begin{aligned}
&\mid X_t^N - \widehat{X}_t^N \mid^2 \\
\leq\ & 4T\int_{t_n}^{t}\Big| b\left(s, \widehat{X}_s^N, \int_0^s f_1(s,u,\widehat{X}_u^N)du, \int_0^s f_2(s,u,\widehat{X}_u^N)du, \ldots,\right. \\
& \left.\int_0^s f_p(s,u,\widehat{X}_u^N)du\right)\Big|^2 ds + 4\Big|\int_{t_n}^{t}\sigma_1\left(s, \widehat{X}_s^N, \int_0^s g_1(s,u,\widehat{X}_u^N)du,\right. \\
& \left.\int_0^s g_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s g_p(s,u,\widehat{X}_u^N)du\right) dW(s)\Big|^2 \\
+\ & 4\Big|\alpha\int_0^{t_n}\left[(t-s)^{\alpha-1}-(t_n-s)^{\alpha-1}\right]\sigma_2\left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du,\right. \\
& \left.\int_0^s h_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s h_p(s,u,\widehat{X}_u^N)du\right) ds\Big|^2 \\
+\ & 4\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_{t_n}^{t}\Big|\sigma_2\left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du, \int_0^s h_2(s,u,\widehat{X}_u^N)du,\ t\right. \\
& \left.\ldots, \int_0^s h_p(s,u,\widehat{X}_u^N)du\right)\Big|^2 ds.
\end{aligned}$$

Let

$$\begin{aligned}
\Gamma = \Big|\int_0^{t_n}&\left[(t-s)^{\alpha-1}-(t_n-s)^{\alpha-1}\right]\sigma_2\left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du,\right. \\
&\left.\int_0^s h_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s h_p(s,u,\widehat{X}_u^N)du\right) ds\Big|^2.
\end{aligned}$$

After some calculations inspired from [**?**, 10] we obtain the existence of two constants $C > 0$ and $0 < \beta < \alpha$ such that

$$\begin{aligned}
\Gamma \leq C\,(t-t_n)^{2(\alpha-\beta)}\int_0^{t_n}&\Big|\sigma_2\left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du,\right. \\
&\left.\int_0^s h_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s h_p(s,u,\widehat{X}_u^N)du\right)\Big|^2 ds.
\end{aligned}$$

By typing mathematical expectation and Itô isometry we have

$$
\begin{aligned}
&\mathbb{E} \mid X_t^N - \widehat{X}_t^N \mid^2 \\
\leq\ & 4T \int_{t_n}^t \mathbb{E}\Big| b\left(s, \widehat{X}_s^N, \int_0^s f_1(s,u,\widehat{X}_u^N)du, \int_0^s f_2(s,u,\widehat{X}_u^N)du, \ldots, \right. \\
& \left. \int_0^s f_p(s,u,\widehat{X}_u^N)du\right) \Big|^2 ds + 4\int_{t_n}^t \mathbb{E}\Big|\sigma_1\left(s, \widehat{X}_s^N, \int_0^s g_1(s,u,\widehat{X}_u^N)du, \right. \\
& \left. \int_0^s g_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s g_p(s,u,\widehat{X}_u^N)du\right) \Big|^2 ds \\
+\ & 4\alpha^2 C\,(t-t_n)^{2(\alpha-\beta)} \int_0^T \mathbb{E}\Big|\sigma_2\left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du, \int_0^s h_2(s,u,\widehat{X}_u^N)du, \right. \\
& \left. \ldots, \int_0^s h_p(s,u,\widehat{X}_u^N)du\right) |^2 ds + 4\alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1} \int_{t_n}^t \mathbb{E}\Big|\sigma_2\left(s, \widehat{X}_s^N, \int_0^s h_1(s,u,\widehat{X}_u^N)du, \right. \\
& \left. \int_0^s h_2(s,u,\widehat{X}_u^N)du, \ldots, \int_0^s h_p(s,u,\widehat{X}_u^N)du\right) \Big|^2 ds,
\end{aligned}
$$

the linear growth imply

$$
\begin{aligned}
\mathbb{E} \mid X_t^N - \widehat{X}_t^N \mid^2 \ \leq\ & 4T\left(1+\sum_{i=1}^p k_i^2\right)\int_{t_n}^t \left(1+\mathbb{E}\mid \widehat{X}_s^N \mid^2\right) ds \\
+\ & 4\left(1+\sum_{i=1}^p k_i^2\right)\int_{t_n}^t \left(1+\mathbb{E}\mid \widehat{X}_s^N \mid^2\right) ds \\
+\ & 4\alpha^2 C\,(t-t_n)^{2(\alpha-\beta)}\left(1+\sum_{i=1}^p k_i^2\right)\int_0^T \left(1+\mathbb{E}\mid \widehat{X}_s^N \mid^2\right) ds \\
+\ & 4\alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1}\left(1+\sum_{i=1}^p k_i^2\right)\int_{t_n}^t \left(1+\mathbb{E}\mid \widehat{X}_s^N \mid^2\right) ds,
\end{aligned}
$$

from lemma 4.1 we have

$$
\begin{aligned}
&\mathbb{E} \mid X_t^N - \widehat{X}_t^N \mid^2 \\
\leq\ & 4T\left(1+\sum_{i=1}^p k_i^2\right)\int_{t_n}^t (1+C)\,ds + 4\left(1+\sum_{i=1}^p k_i^2\right)\int_{t_n}^t (1+C)\,ds \\
+\ & 4\alpha^2 C\,(t-t_n)^{2(\alpha-\beta)}\left(1+\sum_{i=1}^p k_i^2\right)\int_0^T (1+C)\,ds
\end{aligned}
$$

$$\begin{aligned}
&+ \quad 4\alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1}\left(1+\sum_{i=1}^{p} k_i^2\right)\int_{t_n}^{t}(1+C)\,ds \\
&\leq \quad \left[4\left(1+\sum_{i=1}^{p} k_i^2\right)(1+C)\left(T+1+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\right](t-t_n) \\
&+ \quad 4\alpha^2 TC\,(t-t_n)^{2(\alpha-\beta)}.
\end{aligned}$$

By setting

$$C_1 = 4\left(1+\sum_{i=1}^{p} k_i^2\right)(1+C)\left(T+1+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right) \quad and \quad C_2 = 4\alpha^2 TC$$

and noticing that for all $t \in [t_n, t_{n+1}[$, $t - t_n \leq \frac{T}{N}$, we get

$$\mathbb{E} \mid X_t^N - \widehat{X}_t^N \mid^2 \leq \frac{C_1}{N} + \left(\frac{C_2}{N}\right)^{2(\alpha-\beta)}.$$

So, we show there exist two constants $0 < \gamma \leq 1$ and $C > 0$ independent of $N \geq 1$ such that

$$\mathbb{E} \mid X_t^N - \widehat{X}_t^N \mid^2 \leq \frac{C}{N^\gamma}.$$

□

Now, we can prove Theorem 4.1.

Proof. **Existence:** We first show that (X_t^N) is a Cauchy sequence and converges uniformly for every $t \in [0,T]$ to a limit. Let $M \geq N \geq 1$.

Step 1 For $i = 1, 2, \ldots, p$ and $\Theta \in \{M, N\}$, let

$$F_i^\Theta = \int_0^s f_i(s,u,X_u^\Theta)du, \quad G_i^\Theta = \int_0^s g_i(s,u,X_u^\Theta)du, \quad H_i^\Theta = \int_0^s h_i(s,u,X_u^\Theta)du,$$

and

$$\widehat{F}_i^\Theta = \int_0^s f_i(s,u,\widehat{X}_u^\Theta)du, \quad \widehat{G}_i^\Theta = \int_0^s g_i(s,u,\widehat{X}_u^\Theta)du, \quad \widehat{H}_i^\Theta = \int_0^s h_i(s,u,\widehat{X}_u^\Theta)du.$$

First notice that according to the algebraic inequality

$$\mid X_t^M - X_t^N \mid^2 \leq 9\left(I_{1,1}^2 + I_{1,2}^2 + I_{1,3}^2 + I_{2,1}^2 + I_{2,2}^2 + I_{2,3}^2 + I_{3,1}^2 + I_{3,2}^2 + I_{3,3}^2\right) \tag{4.5}$$

where

$$
\begin{aligned}
I^2_{1,1} &= T\int_0^t \left|b\left(s,\widehat{X}^M_s,\widehat{F}^M_1,\dots,\widehat{F}^M_p\right)-b\left(s,X^M_s,F^M_1,\dots,F^M_p\right)\right|^2 ds\\
I^2_{1,2} &= T\int_0^t \left|b\left(s,X^M_s,F^M_1,\dots,F^M_p\right)-b\left(s,X^N_s,F^N_1,\dots,F^N_p\right)\right|^2 ds\\
I^2_{1,3} &= T\int_0^t \left|b\left(s,X^N_s,F^N_1,\dots,F^N_p\right)-b\left(s,\widehat{X}^N_s,\widehat{F}^N_1,\dots,\widehat{F}^N_p\right)\right|^2 ds\\
I^2_{2,1} &= \left|\int_0^t \sigma_1\left(s,\widehat{X}^M_s,\widehat{G}^M_1,\dots,\widehat{G}^M_p\right)-\sigma_1\left(s,X^M_s,G^M_1,\dots,G^M_p\right)dW_s\right|^2\\
I^2_{2,2} &= \left|\int_0^t \sigma_1\left(s,X^M_s,G^M_1,\dots,G^M_p\right)-\sigma_1\left(s,X^N_s,G^N_1,\dots,G^N_p\right)dW_s\right|^2\\
I^2_{2,3} &= \left|\int_0^t \sigma_1\left(s,X^N_s,G^N_1,\dots,g^N_p\right)-\sigma_1\left(s,\widehat{X}^N_s,\widehat{G}^N_1,\dots,\widehat{G}^N_p\right)dW_s\right|^2\\
I^2_{3,1} &= \frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\int_0^t \left|\sigma_2\left(s,\widehat{X}^M_s,\widehat{H}^M_1,\dots,\widehat{H}^M_p\right)-\sigma_2\left(s,X^M_s,H^M_1,\dots,H^M_p\right)\right|^2 ds\\
I^2_{3,2} &= \frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\int_0^t \left|\sigma_2\left(s,X^M_s,H^M_1,\dots,H^M_p\right)-\sigma_2\left(s,X^N_s,H^N_1,\dots,H^N_p\right)\right|^2 ds\\
I^2_{3,3} &= \frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\int_0^t \left|\sigma_2\left(s,X^N_s,H^N_1,\dots,H^N_p\right)-\sigma_2\left(s,\widehat{X}^N_s,\widehat{H}^N_1,\dots,\widehat{H}^N_p\right)\right|^2 ds
\end{aligned}
$$

Step 2: Intermediate calculations.
By using the assymption (A_2) and the lemma 4.2 we have

$$
\begin{aligned}
\mathbb{E}I^2_{1,1} &\le T\int_0^t \left(L\rho\left(\mathbb{E}\mid \widehat{X}^M_s-X^M_s\mid^2\right)+\delta_1\sum_{i=1}^p L_i^2\mathbb{E}\mid \widehat{X}^M_s-X^M_s\mid^2\right)ds\\
&\le T\int_0^t \left(L\rho\left(\frac{C}{M^\gamma}\right)+\delta_1\sum_{i=1}^p L_i^2\frac{C}{M^\gamma}\right)ds\\
&\le T\left(L\vee\delta_1\sum_{i=1}^p L_i\right)\int_0^t\left(\rho\left(\frac{C}{M^\gamma}\right)+\frac{C}{M^\gamma}\right)ds\\
&\le c_1\left(\rho\left(\frac{C}{M^\gamma}\right)+\frac{C}{M^\gamma}\right):=c_1\varrho\left(\frac{C}{M^\gamma}\right)
\end{aligned}
$$

where $c_1=\left(L\vee\delta_1\sum_{i=1}^p L_i\right)T^2$ and ϱ is defined by

$$\varrho(x)=\rho(x)+x. \tag{4.6}$$

Similarly, we have

$$\mathbb{E}I_{1,3}^2 \leq c_1 \varrho\left(\frac{C}{N^\gamma}\right).$$

Also, we obtain

$$\begin{aligned}
\mathbb{E}I_{1,2}^2 &\leq T\int_0^t \left(L\rho\left(\mathbb{E} \mid X_s^M - X_s^N \mid^2\right) + \delta_1 \sum_{i=1}^p L_i^2 \mathbb{E} \mid X_s^M - X_s^N \mid^2\right) ds \\
&\leq T(L \vee \delta_1 \sum_{i=1}^p L_i^2) \int_0^t \left(\rho\left(\mathbb{E} \mid X_s^M - X_s^N \mid^2\right) + \mathbb{E} \mid X_s^M - X_s^N \mid^2\right) ds \\
&\leq c_2 \int_0^t \varrho\left(\mathbb{E} \mid X_s^M - X_s^N \mid^2\right) ds
\end{aligned}$$

where $c_2 = (L \vee \delta_1 \sum_{i=1}^p L_i^2)T$.

Similarly there exist positive constants c_3 and c_4 such that we have

$$\mathbb{E}I_{2,1}^2 \leq c_3 \varrho\left(\frac{C}{M^\gamma}\right), \ \mathbb{E}I_{2,3}^2 \leq c_3 \varrho\left(\frac{C}{N^\gamma}\right), \ \mathbb{E}I_{2,2}^2 \leq c_3 \int_0^t \varrho\left(\mathbb{E} \mid X_s^M - X_s^N \mid^2\right) ds$$

and

$$\mathbb{E}I_{3,1}^2 \leq c_4 \varrho\left(\frac{C}{M^\gamma}\right), \ \mathbb{E}I_{3,3}^2 \leq c_4 \varrho\left(\frac{C}{N^\gamma}\right), \ \mathbb{E}I_{3,2}^2 \leq c_4 \int_0^t \varrho\left(\mathbb{E} \mid X_s^M - X_s^N \mid^2\right) ds.$$

Step 3: Plugging these inequalities in (4.5), we derive that (after taking mathematical expectation)

$$\mathbb{E} \mid X_t^M - X_t^N \mid^2 \leq c_1' \varrho\left(\frac{C}{M^\gamma}\right) + c_1' \varrho\left(\frac{C}{N^\gamma}\right) + c_2' \int_0^t \varrho\left(\mathbb{E} \mid X_s^M - X_s^N \mid^2\right) ds,$$

where $c_1' = 9(c_1 + c_3 + c_4)$ and $c_2' = 9(c_2 + c_3 + c_4)$. Applying the Bihari's lemma, we can write

$$\mathbb{E}\left(\mid X_t^M - X_t^N \mid^2\right) \leq G^{-1}\left[G\left(c_1' \varrho\left(\frac{C}{M^\gamma}\right) + c_1' \varrho\left(\frac{C}{N^\gamma}\right)\right) + c_2' T\right],$$

where $G\left(c_1' \varrho\left(\frac{C}{M^\gamma}\right) + c_1' \varrho\left(\frac{C}{N^\gamma}\right)\right) + c_2' T \in Dom(G^{-1})$.

The assumption (A_2) indicates that $G^{-1}(x) \longrightarrow 0$ as $x \to -\infty$, and we deduct

$$\mathbb{E}\left(\mid X_t^M - X_t^N \mid^2\right) \longrightarrow 0 \quad as \quad M, N \to +\infty$$

Then (X_t^N) is a Cauchy sequence and has a limit X_t in $L^2(J, \mathbb{R}^p)$. From where

$$\lim_{N \to +\infty} \mathbb{E}\left(\mid X_t - X_t^N \mid^2\right) = 0. \tag{4.7}$$

Next, we shall prove that the limit X_t is a solution to equation (4.1). The triangle inequality, equation (4.7) and lemma 4.2 imply

$$\lim_{N\to+\infty} \mathbb{E}\left(\mid X_t - \widehat{X}_t^N \mid^2 \right) = 0.$$

Therefore, as $N \to +\infty$ in (4.4), by the Cauchy–Schwarz inequality and Itô isometry, it follows that

$$\begin{aligned} & X_t = X_0 \\ + & \int_0^t b\left(s, X_s, \int_0^s f_1(s,u,X_u)du, \int_0^s f_2(s,u,X_u)du, \ldots, \int_0^s f_p(s,u,X_u)du\right) ds \\ + & \int_0^t \sigma_1\left(s, X_s, \int_0^s g_1(s,u,X_u)du, \int_0^s g_2(s,u,X_u)du, \ldots, \int_0^s g_p(s,u,X_u)du\right) dW(s) \\ + & \int_0^t (t-s)^{\alpha-1}\sigma_2\left(s, X_s, \int_0^s h_1(s,u,X_u)du, \int_0^s h_2(s,u,X_u)du, \ldots, \right. \\ & \left. \int_0^s h_p(s,u,X_u)du\right) ds. \end{aligned}$$

That is to say, the limit X_t is a solution to equation (4.1).

Uniqueness: Let X_t and Y_t be two solutions of (3.1) with $X_0 = Y_0$. Then X_t and Y_t are two solutions of (4.1). It is obvious that

$$\begin{aligned} & \mathbb{E} \mid X_t - Y_t \mid^2 \\ \leq & \ 3T \int_0^t \mathbb{E}\Big| b\left(s, X_s, \int_0^s f_1(s,u,X_u)du, \ldots, \int_0^s f_p(s,u,X_u)du\right) \\ - & \ b\left(s, Y_s, \int_0^s f_1(s,u,Y_u)du, \ldots, \int_0^s f_p(s,u,X_u)du\right) \Big|^2 ds \\ + & \ 3\int_0^t \mathbb{E}\Big|\sigma_1\left(s, X_s, \int_0^s g_1(s,u,X_u)du, \ldots, \int_0^s g_p(s,u,X_u)du\right) \\ - & \ \sigma_1\left(s, Y_s, \int_0^s g_1(s,u,Y_u)du, \ldots, \int_0^s g_p(s,u,Y_u)du\right) \Big|^2 ds \\ + & \ 3\frac{\alpha^2 T^{2\alpha-1}}{2\alpha-1} \int_0^t \mathbb{E}\Big|\sigma_2(s, X_s, \int_0^s g_1(s,u,X_u)du, \ldots, \int_0^s g_p(s,u,X_u)du) \\ - & \ \sigma_2(s, Y_s, \int_0^s g_1(s,u,Y_u)du, \ldots, \int_0^s g_p(s,u,Y_u)du)\Big|^2 ds \end{aligned}$$

by applying the hypothesis and doing the same calculus as in the existence result we have:

$$\mathbb{E} \mid X_t - Y_t \mid^2 \leq c_5 \int_0^t \varrho \left(\mathbb{E} \mid X_t - Y_t \mid^2\right) ds,$$

where $c_5 = (c_3 \vee c_4)$ such that

$$c_3 = 3T(2C \vee 2\delta_1 \sum_{i=1}^{p} L_i^2) \quad \text{and} \quad c_4 = 3(C\frac{\alpha^2 T^{2\alpha-1}}{2\alpha - 1} \vee \delta_1 \frac{T^{2\alpha-1}}{2\alpha - 1} \sum_{i=1}^{p} L_i^2).$$

Finally Bihary's lemma give,

$$\mathbb{E} \mid X_t - Y_t \mid^2 = 0, \quad t \in [0, T].$$

Then $X_t = Y_t$ for all $t \in [0, T]$ almost surely. Hence the uniqueness.

□

4.2. **Modified Euler Maruyama scheme.** We have previously shown the existence and uniqueness with the euler method, with a constant time step $h = \frac{T}{N}$, we give the simplified version of Euler's scheme.

$$\begin{aligned} X_{t_n} &= X_0 + \int_0^{t_n} b\left(s, X_s, \int_0^s f_1(s,u,X_u)du, \int_0^s f_2(s,u,X_u)du, \dots, \right. \\ &\left. \int_0^s f_p(s,u,X_u)du\right) ds + \int_0^{t_n} \sigma_1 \left(s, X_s, \int_0^s g_1(s,u,X_u)du, \right. \\ &\left. \int_0^s g_2(s,u,X_u)du, \dots, \int_0^s g_p(s,u,X_u)du\right) dW(s) \\ &+ \int_0^{t_n} (t_n - s)^{\alpha-1} \sigma_2 \left(s, X_s, \int_0^s h_1(s,u,X_u)du, \int_0^s h_2(s,u,X_u)du, \dots, \right. \\ &\left. \int_0^s h_p(s,u,X_u)du\right) ds. \end{aligned}$$

By using the subdivision

$$\begin{aligned}
X_{t_n} &= X_0 + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} b\left(t_i, X_{t_i}, \int_0^{t_i} f_1(t_i,u,X_u)du, \int_0^{t_i} f_2(t_i,u,X_u)du, \ldots,\right. \\
&\left. \int_0^{t_i} f_p(t_i,u,X_u)du\right) ds + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sigma_1\left(t_i, X_{t_i}, \int_0^{t_i} g_1(t_i,u,X_u)du,\right. \\
&\left. \int_0^{t_i} g_2(t_i,u,X_u)du, \ldots, \int_0^{t_i} g_p(t_i,u,X_u)du\right) dW(s) \\
&+ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_n - t_i)^{\alpha-1} \sigma_2\left(t_i, X_{t_i}, \int_0^{t_i} h_1(t_i,u,X_u)du, \int_0^{t_i} h_2(t_i,u,X_u)du,\right. \\
&\left. \ldots, \int_0^{t_i} h_p(t_i,u,X_u)du\right) ds \\
&= X_0 + h\sum_{i=0}^{n-1} b\left(t_i, X_{t_i}, \int_0^{t_i} f_1(t_i,u,X_u)du, \int_0^{t_i} f_2(t_i,u,X_u)du,\right. \\
&\left. \ldots, \int_0^{t_i} f_p(t_i,u,X_u)du\right) + \sum_{i=0}^{n-1} \sigma_1\left(t_i, X_{t_i}, \int_0^{t_i} g_1(t_i,u,X_u)du,\right. \\
&\left. \int_0^{t_i} g_2(t_i,u,X_u)du, \ldots, \int_0^{t_i} g_p(t_i,u,X_u)du\right) \Delta W_i \\
&+ h\sum_{i=0}^{n-1} (t_n - t_i)^{\alpha-1} \sigma_2\left(t_i, X_{t_i},\right. \\
&\left. \int_0^{t_i} h_1(t_i,u,X_u)du, \int_0^{t_i} h_2(t_i,u,X_u)du, \ldots, \int_0^{t_i} h_p(t_i,u,X_u)du\right)
\end{aligned}$$

where $\Delta W_i = W_{t_{i+1}} - W_{t_i}$ is the increment of Brownian motion W_t. Let's define

$$\underline{s} = t_n, \quad s \in [t_n, t_{n+1}[\quad \text{and} \quad \widehat{X}_t = \sum_{n=0}^{N} X_{t_n} \mathbb{1}_{[t_n, t_{n+1}[}(t), t \in [0,T]$$

Then, the modified scheme follows

$$X_t^h = X_0 + \int_0^t b\left(\underline{s}, \widehat{X}_s, \int_0^s f_1(s,u,\widehat{X}_u)du, \int_0^s f_2(s,u,\widehat{X}_u)du, \ldots,\right.$$

$$
\begin{aligned}
&+ \quad \int_0^s f_p(s,u,\widehat{X}_u)du\Big) ds \int_0^t \sigma_1\left(\underline{s}, \widehat{X}_s, \int_0^s g_1(s,u,\widehat{X}_u)du, \int_0^s g_2(s,u,\widehat{X}_u)du,\right.\\
(4.8) \quad &\left.\ldots, \int_0^s g_p(s,u,\widehat{X}_u)du\right) dW(s)\\
&+ \quad \int_0^t (t-\underline{s})^{\alpha-1}\sigma_2\left(\underline{s}, \widehat{X}_s, \int_0^s h_1(s,u,\widehat{X}_u)du, \int_0^s h_2(s,u,\widehat{X}_u)du, \ldots,\right.\\
&\qquad \left.\int_0^s h_p(s,u,\widehat{X}_u)du\right) ds.
\end{aligned}
$$

The proofs of the following two lemmas are based respectively on those of lemmas 4.1 and 4.2.

Lemma 4.3. *Under the assumptions (A_1),(A_2) and (A_3), there exist a positive constant C independent of h such that*

$$\mathbb{E} \mid \widehat{X}_t \mid^2 \leq C \quad \textit{and} \quad \mathbb{E} \mid X_t^h \mid^2 \leq C \quad \forall t \in [0,T].$$

Lemma 4.4. *Under the assumptions (A_1),(A_2)and (A_3), there exist positive constants C independent of h and $0 < \gamma \leq 1$ such that*

$$\mathbb{E} \mid X_t^h - \widehat{X}_t \mid^2 \leq Ch^\gamma \quad \forall t \in [0,T].$$

We give the main result in this section.

Theorem 4.2. *Under the assumptions (A_1),(A_2) and (A_3), we have*

$$\lim_{h\to 0} \mathbb{E} \mid X_t - X_t^h \mid^2 = 0, \quad \forall t \in [0,T].$$

Proof. As before, let's proceed step by step. For $i = 1, 2, \ldots, p$, let

$$F_i = \int_0^s f_i(s,u,X_u)du, \quad G_i = \int_0^s g_i(s,u,X_u)du, \quad H_i = \int_0^s h_i(s,u,X_u)du,$$

$$\widehat{F}_i = \int_0^s f_i(s,u,\widehat{X}_u)du, \quad \widehat{G}_i = \int_0^s g_i(s,u,\widehat{X}_u)du, \quad \widehat{H}_i = \int_0^s h_i(s,u,\widehat{X}_u)du,$$

and

$$F_i^h = \int_0^s f_i(s,u,X_u^h)du, \quad G_i^h = \int_0^s g_i(s,u,X_u^h)du, \quad H_i^h = \int_0^s h_i(s,u,X_u^h)du.$$

Step 1

$$
\begin{aligned}
J_{1,1} &= \int_0^t \left| b\left(s, X_s, F_1, \ldots, F_p\right) - b\left(\underline{s}, X_s, F_1, \ldots, F_p\right) \right| ds,\\
J_{1,2} &= \int_0^t \left| b\left(\underline{s}, X_s, F_1, \ldots, F_p\right) - b\left(\underline{s}, X_s^h, F_1^h, \ldots, F_p^h\right) \right| ds,\\
J_{1,3} &= \int_0^t \left| b\left(\underline{s}, X_s^h, F_1^h, \ldots, F_p^h\right) - b\left(\underline{s}, \widehat{X}_s, \widehat{F}_1, \ldots, \widehat{F}_p\right) \right| ds.
\end{aligned}
$$

$$
\begin{aligned}
J_{2,1} &= \Big| \int_0^t \sigma_1\left(s, X_s, G_1, \ldots, G_p\right) - \sigma_2\left(\underline{s}, X_s, G_1, \ldots, G_p\right) dW_s \Big|,\\
J_{2,2} &= \Big| \int_0^t \sigma_2\left(\underline{s}, X_s, G_1, \ldots, G_p\right) - \sigma_2\left(\underline{s}, X_s^h, G_1^h, \ldots, G_p^h\right) dW_s \Big|,\\
J_{2,3} &= \Big| \int_0^t \sigma_2\left(\underline{s}, X_s^h, G_1^h, \ldots, G_p^h\right) - \sigma_2\left(\underline{s}, \widehat{X}_s, \widehat{G}_1, \ldots, \widehat{G}_p\right) dW_s \Big|.
\end{aligned}
$$

$$
\begin{aligned}
J_{3,1} &= T^{\alpha-1} \int_0^t \left| \sigma_2\left(s, X_s, H_1, \ldots, H_p\right) - \sigma_2\left(\underline{s}, X_s, H_1, \ldots, H_p\right) \right| ds,\\
J_{3,2} &= T^{\alpha-1} \int_0^t \left| \sigma_2\left(\underline{s}, X_s, H_1, \ldots, H_p\right) - \sigma_2\left(\underline{s}, X_s^h, H_1^h, \ldots, H_p^h\right) \right| ds,\\
J_{3,3} &= T^{\alpha-1} \int_0^t \left| \sigma_2\left(\underline{s}, X_s^h, H_1^h, \ldots, H_p^h\right) - \sigma_2\left(\underline{s}, \widehat{X}_s, \widehat{H}_1, \ldots, \widehat{H}_p\right) \right| ds.
\end{aligned}
$$

Step 2: By using assumption (A_3), we can write

$$
\begin{aligned}
\mathbb{E}\left(J_{1,1}^2\right) &\leq \int_0^t \mathbb{E}\left| b\left(s, X_s, F_1, \ldots, F_p\right) - b\left(\underline{s}, X_s, F_1, \ldots, F_p\right) \right|^2 ds\\
&\leq \widetilde{k} h T \int_0^t \left(1 + \mathbb{E} \mid X_s \mid^2\right) ds \leq \widetilde{k} h T \int_0^t (1 + C)\, ds \leq C_1 h,
\end{aligned}
$$

where $C_1 = (1 + C)\, \widetilde{k} T^2$. Applying Cauchy Schwarz inequality we obtain by assumption (A_2):

$$
\begin{aligned}
\mathbb{E}\left(J_{1,2}^2\right) &\leq \int_0^t \mathbb{E}\left| b\left(\underline{s}, X_s, F_1, \ldots, F_p\right) - b\left(\underline{s}, X_s^h, F_1^h, \ldots, F_p^h\right) \right|^2 ds\\
&\leq \int_0^t \left(C\rho\left(\mathbb{E} \mid X_s - X_s^h \mid^2\right) + \delta_1 \sum_{i=1}^p L_i^2 \mathbb{E} \mid X_s - X_s^h \mid^2 \right) ds
\end{aligned}
$$

$$\begin{aligned}
&\leq \left(C \vee \delta_1 \sum_{i=1}^{p} L_i^2\right) \int_0^t \left(\rho\left(\mathbb{E} \mid X_s - X_s^h \mid^2\right) + \mathbb{E} \mid X_s - X_s^h \mid^2\right) ds \\
&\leq C_2 \int_0^t \varrho\left(\mathbb{E} \mid X_s - X_s^h \mid^2\right) ds,
\end{aligned}$$

where $C_2 = (C \vee \delta_1 \sum_{i=1}^{p} L_i^2)$ and ϱ is given by (4.6). Also,

$$\begin{aligned}
\mathbb{E}\left(J_{1,3}^2\right) &\leq \int_0^t \mathbb{E}\left| b\left(\underline{s}, X_s^h, F_1^h, \ldots, F_p^h\right) - b\left(\underline{s}, \widehat{X}_s, \widehat{F}_1, \ldots, \widehat{F}_p\right)\right|^2 ds \\
&\leq \int_0^t \left(C\rho\left(\mathbb{E} \mid X_s^h - \widehat{X}_s \mid^2\right) + \delta_1 \sum_{i=1}^{p} L_i^2 \mathbb{E} \mid X_s^h - \widehat{X}_s \mid^2 \right) ds \\
&\leq \int_0^t \left(C\rho\left(Ch^\gamma\right) + \delta_1 Ch^\gamma \sum_{i=1}^{p} L_i^2 \right) ds \\
&\leq \left(C \vee \delta_1 \sum_{i=1}^{p} L_i^2 \right) \int_0^t \left(\rho\left(Ch^\gamma\right) + Ch^\gamma\right) ds \\
&\leq C_3\left(\rho\left(Ch^\gamma\right) + Ch^\gamma\right) := C_3 \varrho\left(Ch^\gamma\right)
\end{aligned}$$

where $C_3 = (C \vee \delta_1 \sum_{i=1}^{p} L_i^2) T$.

Similarly we also have,

$$\mathbb{E}\left(J_{2,1}^2\right) \leq C_1 h,\ \mathbb{E}\left(J_{2,2}^2\right) \leq C_2 \int_0^t \varrho\left(\mathbb{E} \mid X_s - X_s^h \mid^2\right) ds \quad \mathbb{E}\left(J_{2,3}^2\right) \leq C_3 \varrho\left(Ch^\gamma\right),$$

$$\mathbb{E}\left(J_{3,1}^2\right) \leq \widetilde{C}_1 h,\ \mathbb{E}\left(J_{3,2}^2\right) \leq \widetilde{C}_2 \int_0^t \varrho\left(\mathbb{E} \mid X_s - X_s^h \mid^2\right) ds, \quad \mathbb{E}\left(J_{3,3}^2\right) \leq \widetilde{C}_3 \varrho\left(Ch^\gamma\right),$$

where

$$\widetilde{C}_1 = C_1 T^{2\alpha-2}, \quad \widetilde{C}_2 = C_2 T^{2\alpha-2}, \quad \widetilde{C}_3 = C_3 T^{2\alpha-2}.$$

Step 3: So

$$\begin{aligned}
&\mathbb{E} \mid X_t - X_t^h \mid^2 \leq 9\left[h(2C_1 + \widetilde{C}_1)\right. \\
&+ \left.(2C_2 + \widetilde{C}_2) \int_0^t \varrho\left(\mathbb{E} \mid X_s - X_s^h \mid^2\right) ds + (2C_3 + \widetilde{C}_3) \varrho\left(Ch^\gamma\right)\right] \\
&\leq C_1' h + C_3' \varrho\left(Ch^\gamma\right) + C_2' \int_0^t \varrho\left(\mathbb{E} \mid X_s - X_s^h \mid^2\right) ds \qquad (4.9)
\end{aligned}$$

where

$$C'_1 = 9(2C_1 + \widetilde{C}_1), \quad C'_2 = 9(2C_2 + \widetilde{C}_2), \quad C'_3 = 9(2C_3 + \widetilde{C}_3),$$

We apply the Bihari inequality, and we can write

$$\mathbb{E}\left(| X_s - X_s^h |^2\right) \leq G^{-1}\left[G\left(C'_1 h + C'_3 \varrho\left(Ch^\gamma\right)\right) + C'_2 T\right],$$

where $G\left(C'_1 h + C'_3 \varrho\left(Ch^\gamma\right)\right) + C'_2 T \in Dom\left(G^{-1}\right)$ and as $\int_{0^+} \frac{ds}{\rho(s)} = +\infty$ indicating that $G\left(C'_1 h + C'_3 \varrho\left(Ch^\gamma\right)\right) + C'_2 T \longrightarrow -\infty$ as $h \to 0$, $]-\infty, 0] \subset Dom\left(G^{-1}\right)$, and $G^{-1}(x) \longrightarrow 0$ as $x \to -\infty$, we conclude that

$$\mathbb{E}\left(| X_s - X_s^h |^2\right) \longrightarrow 0 \quad h \to 0.$$

□

Theorem 4.3. *Under the same assumptions of the theorem* (4.1)*, if* $\rho(u) = u$*, for* $u \geqslant 0$*. There exist a positive constant C independent of h and* $0 < \gamma \leq 1$ *such that*

$$\mathbb{E}\left(| X_s - X_s^h |^2\right) \leqslant Ch^\gamma \quad \forall t \in [0, T].$$

Proof. Since $\rho(u) = u$, for $u \geqslant 0$ we can rewrite the estimate (4.9) as

$$\mathbb{E} | X_t - X_t^h |^2 \leq C_4\left(Ch^\gamma + h\right) + C'_2 \int_0^t \mathbb{E} | X_s - X_s^h |^2,$$

Finally, the Gronwall inequality completes the proof. □

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