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COMPARATIVE STUDY OF THE LAPLACE-ADOMIAN METHOD AND THE VARIATIONAL ITERATIONS METHOD FOR DETERMINING THE EXACT SOLUTIONS OF SOME PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This work is a verification of the effectiveness of Laplace-Adomian and variational iterations methods for solving partial differential equations. The coupling of Laplace and Adomian methods has made it possible to exploit Adomian polynomials with Laplace transforms, as well as their inverse, to overcome the difficulties associated with the non-linearity of the equations. The variational iteration method, with its correction functional, facilitates the determination of the general Lagrange multiplier, which is essential for the rest of the solution. These methods have enabled us to obtain the exact solutions.

1. INTRODUCTION

Mathematical modelling of physical systems leads to functional equations (ordinary differential equations (ODE), partial differential equations (PDE), integrals and integrals, etc.). Several numerical analysis methods have been developed to solve these types of equations, such as approximation and analytical methods [4, 17].

In this work, we focus on the partial differential equations. These equations are omnipresent in science, appearing in mechanics, epidemiology and many

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other fields. They are generally non-linear. And it's equally difficult, if not impossible, to find their exact solutions by conventional methods [9, 11]. This difficulty, or impossibility, often leads us to settle for approximate solutions. For this, discretization or linearization methods are used. Nevertheless, in addition to the classical methods, several other methods have been developed to solve these types of equations. These include Laplace-Adomian method and variational iterations method (VIM). These tried-and-tested methods produce efficient algorithms that converge rapidly to exact solutions when they exist. In addition, these methods avoid discretization and linearization. They take into account the initial and boundary conditions of the problems studied. It should be stressed that the main difficulty with the Adomian method lies in the calculation of Adomian polynomials, which can be very tedious for certain types of problem.

On the other hand, Laplace transforms do not allow us to solve non-linear equations, as there is no Laplace transform of non-linear terms. To get around this difficulty, or to compensate for the inadequacy of the Laplace transform, a coupling is made between Laplace transforms and the Adomian decompositional method. The result is the Laplace-Adomian method. The aim of this work is to experiment with mathematical methods avoiding linearization and discretization of space and time to better solve linear and non-linear partial differential equation models.

2. Description of the methods

2.1. The Laplace-Adomian method. The presentation below is shows how the Laplace-Adomian method works [1, 10–13, 16, 18]. It is the algorithm of the method.

Consider a functional equation

$$Au = h. \tag{2.1}$$

with

$$A = L + R + N.$$

The equation (2.1) becomes:

$$Lu + Ru + Nu = h, \tag{2.2}$$

where u is a unknown function of H (H is a Hilbert space), L and R are linear operators; and L reversible, with L^{-1} as reverse. N is a nonlinear operator from a Hilbert space H into H. h is given function in H.

By applying the transform $\mathcal{L}$ of Laplace at the equation (2.2), gives:

$$\mathcal{L}\left(Lu\right)+\mathcal{L}\left(Ru\right)+\mathcal{L}\left(Nu\right)=\mathcal{L}\left(h\right). \tag{2.3}$$

Let's set

$$L_t=\frac{\partial}{\partial t},\qquad L_{tt}=\frac{\partial^2}{\partial t^2},\qquad \text{and}\qquad L_{tt}^{-1}(.)=\int_0^t\int_0^s(.)dsdt;$$

with the initial and boundary conditions, we have the following relation:

$$u(x,0)=f(x)\quad et\quad u_t(x,0)=g(x), \tag{2.4}$$

$$\mathcal{L}\left(Lu(x,t)\right)+\mathcal{L}\left(Ru(x,t)\right)+\mathcal{L}\left(Nu(x,t)\right)=\mathcal{L}\left(h(x,t)\right). \tag{2.5}$$

Finally, with the conditions (2.4) the Laplace transform, into the equation (2.5) this gives:

$$s^2\mathcal{L}(Lu(x,t))-su(x,0)-u_t(x,0)+\mathcal{L}(Ru(x,t))+\mathcal{L}(Nu(x,t))=\mathcal{L}h(x,t). \tag{2.6}$$

This gives the expression:

$$\begin{aligned}\mathcal{L}(u(x,t))=&\frac{1}{s}u(x,0)+\frac{1}{s^2}u_t(x,t)+\frac{1}{s^2}\mathcal{L}(h(x,t))\\&-\frac{1}{s^2}\mathcal{L}(Ru(x,t))-\frac{1}{s^2}\mathcal{L}(Nu(x,t)).\end{aligned} \tag{2.7}$$

Or again

$$\mathcal{L}(u(x,t))=\frac{1}{s}f(x)+\frac{1}{s^2}g(x)+\frac{1}{s^2}\mathcal{L}(h(x,t))-\frac{1}{s^2}\mathcal{L}(Ru(x,t))-\frac{1}{s^2}\mathcal{L}(Nu(x,t)). \tag{2.8}$$

The solution u of the equation (2.1), when it exists, is sought in the form of the série:

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t). \tag{2.9}$$

The non-linear part is also expressed as a series of polynomials:

$$N(u(x,t))=\sum_{n=0}^{\infty}A_n(x,t), \tag{2.10}$$

where A_n are Adomian polynomials defined by the formula. Substituting (2.9) and (2.10) in (2.8) gives the expression:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{L}(u_n(x,t)) &= \frac{1}{s}f(x)+\frac{1}{s^2}g(x)+\frac{1}{s^2}\mathcal{L}(h(x,t))\\ &\quad-\sum_{n=0}^{\infty}\left(\frac{1}{s^2}\mathcal{L}(Ru_n(x,t))+\frac{1}{s^2}\mathcal{L}(A_n(x,t))\right).\end{aligned} \tag{2.11}$$

This leads to the following Laplace-Adomian algorithm:

$$\begin{cases}\mathcal{L}\left(u_0(x,t)\right)=\frac{1}{s}f(x)+\frac{1}{s^2}g(x)+\frac{1}{s^2}\mathcal{L}\left(h(x,t)\right)\\ \mathcal{L}\left(u_{n+1}(x,t)\right)=-\left(\frac{1}{s^2}\mathcal{L}(Ru_n(x,t))+\frac{1}{s^2}\mathcal{L}(A_n(x,t))\right), \quad n\geq 0.\end{cases} \tag{2.12}$$

By applying the inverse of the Laplace transform into the expressions of $u_0(x,t)$ and $u_{n+1}(x,t)$ are established:

$$\begin{cases}u_0(x,t) &= \mathcal{L}^{-1}\left(\frac{f(x)}{s}+\frac{g(x)}{s^2}+\frac{1}{s^2}\mathcal{L}[h(x,t)]\right)\\ u_{n+1}(x,t) &= -\mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}[Ru_n(x,t)]-\frac{1}{s^2}\mathcal{L}[A_n(x,t)]\right), \quad n\geq 0.\end{cases} \tag{2.13}$$

2.2. **The Variational Iterations Method.** The variational iteration method (VIM) was proposed and developed by Chinese mathematician Je-Haun-He in the early 1990s [7]. It was first proposed to solve problems in mechanics. The method has been used to solve a wide variety of linear and non-linear problems, with successive approximations rapidly converging to the exact solution if it exists. The method is based on the determination of the multiplier optimally via variational theory [3, 8, 15].

To illustrate the basic ideas behind this method, consider the following non-linear differential equation:

$$Lu(x,t)+Nu(x,t)=g(x,t), \tag{2.14}$$

with

L: a linear differential operator;
N: a non-linear differential operator;
g: a known function.

According to the following variational iteration method, a functional correction can be constructed in the following way:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s) \left(Lu_n(x,s) + N\tilde{u_n}(x,s) - g(x,s) \right) ds, \tag{2.15}$$

with $s \in [0;t]$ where

λ: is a general Lagrange multiplier;

n: is an index which represents the n^{th} approximation;

$\tilde{u}_n(x,s)$: is considered as a restricted variation; This means that $\delta\tilde{u}_n(x,t) = 0$.

To solve the differential equation (2.14) by the method of variational iterations, we need to determine the Lagrange multiplier λ, which will be be optimally identified by integration by parts.

Then the successive approximations $u_n(x,t)$ of the solution $u(x,t)$ will be obtained by using the Lagrange multiplier λ and a function $u_0(x,t)$ well-chosen (which must at least satisfy the initial conditions). Consequently, the exact solution will be the limit:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.16}$$

Though, so $L = \dfrac{\partial(.)}{\partial t}$, assuming the correction functional is stationary with respect to u_n and posing $\delta\tilde{u_n}(x,t) = 0$, the result will be:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(s) \left(\frac{\partial u_n(x,s)}{\partial s} + N\tilde{u_n}(x,s) - g(x,s) \right) ds, \tag{2.17}$$

either

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(s) \left(\frac{\partial u_n(x,s)}{\partial s} \right) ds = 0. \tag{2.18}$$

Integration by parts of the expression $\displaystyle\int_0^t \delta\lambda(s) \left(\frac{\partial u_n(x,s)}{\partial s} \right) ds$, leads to

$$\int_0^t \delta\lambda(s) \left(\frac{\partial u_n(x,s)}{\partial s} \right) ds = \lambda(t)\delta u_n(x,t) - \int_0^t \lambda^{'} \delta u_n(x,s) ds. \tag{2.19}$$

Substituting the relationship (2.19) into the relationship (2.18) gives:

$$\delta u_{n+1}(x,t) = (1 + \lambda(t))\, \delta u_n(x,t) - \int_0^t \lambda^{'}(s) \delta u_n(x,s) ds = 0. \tag{2.20}$$

Then, from equation (2.20), we derive the following stationarity conditions:

$$\begin{cases} \delta u_n(x,s)ds : \lambda^{'}(s) = 0 \\ \delta u_n(x,s)ds : 1 + \lambda(t) = 0 \end{cases}, \tag{2.21}$$

which gives:

$$\lambda(t) = -1. \tag{2.22}$$

Thus, the correction functional (2.15) becomes

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(Lu_n(x,s) + N\tilde{u_n}(x,s) - g(x,s)\right) ds. \tag{2.23}$$

If $L = \dfrac{\partial^2(.)}{\partial t^2}$, then

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(s) \left(\frac{\partial^2 u_n(x,s)}{\partial s^2}\right) ds = 0. \tag{2.24}$$

Double integration by parts of the expression $\displaystyle\int_0^t \delta\lambda(s) \left(\frac{\partial^2 u_n(x,s)}{\partial s^2}\right) ds$, the following result is obtained

$$\begin{aligned} \int_0^t \delta\lambda(s) \left(\frac{\partial^2 u_n(x,s)}{\partial s^2}\right) ds &= \lambda(t)\delta\frac{\partial u_n(x,t)}{\partial t} - \lambda^{'}\delta u_n(x,t) \\ &+ \int_0^t \lambda^{''}(s)\delta u_n(x,s)ds. \end{aligned} \tag{2.25}$$

Still proceeding by substitution, from the equation (2.25) in the relation (2.24), the expression (2.26) below follows

$$\begin{aligned} \delta u_{n+1}(x,t) = \left(1 - \lambda^{'}(t)\right) \delta u_n(x,t) + \lambda(t)\delta\frac{\partial u_n(x,t)}{\partial t} \\ + \int_0^t \lambda^{''}(s)\delta u_n(x,s)ds = 0. \end{aligned} \tag{2.26}$$

The stationarity conditions are given by the system (2.27) below:

$$\begin{cases} \delta u_n(x,s)ds : \lambda^{''}(s) = 0 \\ \delta u_n(x,s)ds : 1 - \lambda^{'}(t) = 0 \\ u_n^{'}(x,t) : \lambda(t) = 0 \end{cases}, \tag{2.27}$$

which gives:

$$\left\{ \begin{array}{l} \lambda^{''}(s)=0 \quad \Rightarrow \quad \lambda^{'}(s)=1 \quad \Rightarrow \quad \lambda(s)=s+b \\ 1-\lambda^{'}(t)=0 \quad \Rightarrow \quad \lambda^{'}(t)=1 \end{array} \right. . \tag{2.28}$$

So,

$$\left\{ \begin{array}{l} \lambda(t)=0 \quad \Rightarrow \quad t+b=0 \Rightarrow b=-t, \\ \lambda(s)=s-t. \end{array} \right. .$$

Thus, the correction functional (2.15) becomes:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t (s-t)\left(Lu_n(x,s)+N\tilde{u_n}(x,s)-g(x,s)\right)ds. \tag{2.29}$$

If $L=\dfrac{\partial^3}{\partial t^3}$, which gives:

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda(s)\left(\frac{\partial^3 u_n(x,s)}{\partial s^3}\right)ds=0. \tag{2.30}$$

After triple integration by parts of the expression $\int_0^t \delta\lambda(s)\left(\dfrac{\partial^3 u_n(x,s)}{\partial s^3}\right)ds$, this results in:

$$\begin{aligned} \delta u_{n+1}(x,t) &= (1+\lambda^{''}(t))\delta u_n(x,t)+\lambda(t)\delta\frac{\partial^2 u_n(x,t)}{\partial t^2}-\lambda^{'}(t)\delta\frac{\partial u_n(x,t)}{\partial t} \\ &\quad -\int_0^t \delta\lambda^{'''}(s)u_n(x,s)ds=0. \end{aligned} \tag{2.31}$$

Consequently, the stationarity conditions are expressed by the system (2.32) below:

$$\left\{ \begin{array}{l} \delta u_n(x,s)ds : 1+\lambda^{''}(s)=0 \\ \delta u_n^{''}(x,s)ds : \lambda^{'}(t)=0 \\ \delta u_n^{'}(x,t) : \lambda^{'}(t)=0 \\ \delta u_n(x,s) : \lambda^{'''}(s)=0. \end{array} \right. \tag{2.32}$$

So,

$$\left\{ \begin{array}{l} \lambda^{'''}(s)=0 \quad \Rightarrow \quad \lambda^{''}(s)=a \quad \Rightarrow \quad \lambda^{'}(s)=as+b \\ \qquad\qquad\qquad \Rightarrow \lambda(s)=\frac{1}{2}as^2+bs+c \\ 1+\lambda^{''}(s)=0 \quad \Rightarrow \quad \lambda^{''}(s)=-1 \Rightarrow a=-1. \end{array} \right. . \tag{2.33}$$

So,

$$\begin{cases} \lambda'(t) = 0 & \Rightarrow \quad at + b = 0 \Rightarrow b = t \\ \lambda(t) = 0 & \Rightarrow \quad -\frac{1}{2}t^2 + t^2 + c = 0 \quad \Rightarrow \quad c = -\frac{1}{2}t^2. \end{cases} \tag{2.34}$$

The Lagrange multiplier in this case is therefore given by the following expression (2.35):

$$\lambda(s) = \frac{(-1)^3}{2!}(s-t)^2. \tag{2.35}$$

Thus, the correction functional (2.15) becomes:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \frac{(-1)^3}{2!}(s-t)^2 \left(Lu_n(x,s) + N\tilde{u_n}(x,s) - g(x,s)\right) ds. \tag{2.36}$$

In a global sense, $L = \dfrac{\partial^m(.)}{\partial t^m}$, then the result will be

$$\lambda(s) = \frac{(-1)^m}{(m-1)!}(s-t)^{m-1}. \tag{2.37}$$

Once the Lagrangian multiplier $lambda$ is identified, then, by choosing an initial approximation $u_0(x,t)$ of the problem solution (2.14), the other successive approximations $u_i(x,t), \quad i \geq 1$ are easily obtained using the correction functional (2.15). Subsequently, the exact solution $u(x,t)$ is given by:

$$u(x,t) = \lim_{n \to +\infty} u_n(x,t). \tag{2.38}$$

The concept of convergence of the variational iteration method has been studied and demonstrated by several authors [5, 14] for the solution of functional equations. Important theorems have been given involving the necessary and sufficient conditions for convergence [14].

3. Application of the Laplace-Adomian method

3.1. **Problem 1.** Consider the following equation [6]

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} - bu^3(x,t) \\ u(x,0) = \sqrt{\dfrac{2}{b}}\left(\dfrac{2x}{x^2+1}\right) \end{cases} . \tag{3.1}$$

Let's solve this equation using the Laplace-Adomian method. Let:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - bu^3(x,t). \tag{3.2}$$

Applying the Laplace transform to equation (3.2), the result is

$$\mathcal{L}\left(\frac{\partial u(x,t)}{\partial t}\right) = \mathcal{L}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) - b\mathcal{L}\left(u^3(x,t)\right). \tag{3.3}$$

$$s\mathcal{L}u(x,t) = u(x,0) + \mathcal{L}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) - b\mathcal{L}\left(u^3(x,t)\right), \tag{3.4}$$

either

$$\mathcal{L}u(x,t) = \frac{1}{s}u(x,0) + \frac{1}{s}\mathcal{L}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) - \frac{1}{s}b\mathcal{L}(u^3(x,t)). \tag{3.5}$$

Applying the inverse Laplace transform to the relationship (3.5) gives:

$$u(x,t) = \sqrt{\frac{2}{b}}\left(\frac{2x}{x^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(\frac{\partial^2 u(x,t)}{\partial x^2})\right) - b\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(u^3(x,t))\right), \tag{3.6}$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{3.7}$$

and

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n(x,t). \tag{3.8}$$

This gives us the following algorithm:

$$\begin{cases} u_0(x,t) = \sqrt{\frac{2}{b}}\left(\frac{2x}{x^2+1}\right) \\ u_{n+1}(x,t) = \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(\frac{\partial^2 u_n(x,t)}{\partial x^2})\right) - b\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(A_n(x,t))\right) \end{cases}. \tag{3.9}$$

Calculating $A_0(x,t)$ and $u_1(x,t)$ gives:

$$A_0(x,t) = u_0^3(x,t) = \frac{2}{b}\sqrt{\frac{2}{b}}\left(\frac{2x}{x^2+1}\right)^3, \tag{3.10}$$

$$u_1(x,t) = \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(\frac{\partial^2 u_0(x,t)}{\partial x^2})\right) - b\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(A_0(x,t))\right), \tag{3.11}$$

$$u_1(x,t) = \sqrt{\frac{2}{b}}\left(\frac{4x^3-12x}{(x^2+1)^3} - \frac{16x^3}{(x^2+1)^3}\right)\mathcal{L}^{-1}\left(\frac{1}{s^2}\right). \tag{3.12}$$

Now,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t.$$

Thus

$$u_1(x,t) = \sqrt{\frac{2}{b}}\left(\frac{-12xt}{(x^2+1)^3}\right). \tag{3.13}$$

Therefore:

$$u_1(x,t) = u_0(x,t)\left(\frac{-6t}{x^2+1}\right). \tag{3.14}$$

Let's calculate $A_1(x,t)$ et $u_2(x,t)$:

$$A_1(x,t) = 3u_0^2(x,t)u_1(x,t) = -\frac{1}{b}\sqrt{\frac{2}{b}}\left(\frac{288x^3t}{(x^2+1)^4}\right), \tag{3.15}$$

$$u_2(x,t) = \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(\frac{\partial^2 u_1(x,t)}{\partial x^2})\right) - b\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(A_1(x,t))\right). \tag{3.16}$$

Such

$$u_2(x,t) = \sqrt{\frac{2}{b}}\left(\frac{-144x^3+144x}{(x^2+1)^4} + \frac{288x^3}{(x^2+1)^4}\right)\mathcal{L}^{-1}\left(\frac{1}{s^3}\right), \tag{3.17}$$

knowing that

$$\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2!}.$$

Therefore,

$$u_2(x,t) = \frac{1}{2}\sqrt{\frac{2}{b}}\left(\frac{72xt^2}{(x^2+1)^3}\right), \tag{3.18}$$

either

$$u_2(x,t) = u_0(x,t)\left(\frac{36t^2}{(x^2+1)^2}\right). \tag{3.19}$$

Calculating $A_2(x,t)$ and $u_3(x,t)$ gives:

$$A_2(x,t) = 3u_0^2u_2 + 3u_1^2u_0 = \frac{1}{b}\sqrt{\frac{2}{b}}\left(\frac{3456x^3t^2}{(x^2+1)^5}\right), \tag{3.20}$$

$$u_3(x,t) = \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(\frac{\partial^2 u_2(x,t)}{\partial x^2})\right) - b\mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(A_2(x,t))\right), \tag{3.21}$$

either

$$u_3(x,t) = 2\sqrt{\frac{2}{b}}\left(\frac{2160x^3 - 1296x}{(x^2+1)^5} - \frac{3456x^3}{(x^2+1)^5}\right)\mathcal{L}^{-1}\left(\frac{1}{s^4}\right), \tag{3.22}$$

knowing that:

$$\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{t^3}{3!}. \tag{3.23}$$

Thus

$$u_3(x,t) = \frac{1}{3}\sqrt{\frac{2}{b}}\left(\frac{-432xt^3}{(x^2+1)^4}\right), \tag{3.24}$$

either

$$u_2(x,t) = u_0(x,t)\left(\frac{-216t^3}{(x^2+1)^3}\right). \tag{3.25}$$

The solution being in the form:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots, \tag{3.26}$$

such that

$$\begin{aligned} u(x,t) = u_0(x,t) + u_0(x,t)\left(\frac{-6t}{x^2+1}\right) + u_0(x,t)\left(\frac{36t^2}{(x^2+1)^2}\right) \\ + u_0(x,t)\left(\frac{-216t^3}{(x^2+1)^3}\right) + \ldots. \end{aligned} \tag{3.27}$$

The transformation of the terms gives:

$$\begin{aligned} u(x,t) = u_0(x,t)\left[1 + \left(\frac{-6t}{x^2+1}\right) + \left(\frac{-6t}{x^2+1}\right)^2 + \left(\frac{-6t}{x^2+1}\right)^3 y \right. \\ \left. + \left(\frac{-6t}{x^2+1}\right)^4 \ldots + \left(\frac{-6t}{x^2+1}\right)^n + \cdots\right]. \end{aligned} \tag{3.28}$$

This expression is in geometric progression. Consider the n terms of the geometric sequence with first term u_0 and reason $q = \left(\dfrac{-6t}{x^2+1}\right)$, $u(x,t)$ converges if $|q| < 1$,

$$u(x,t) = \lim_{n\to+\infty} u_n(x,t), \tag{3.29}$$

$$u(x,t) = \sqrt{\frac{2}{b}}\left(\frac{2x}{x^2+6t+1}\right). \tag{3.30}$$

Applying the Laplace-Adomian method to our problem led us to its exact solution.

3.2. **Problem 2.** Consider the following equation [7]

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \left(u(x,t)\dfrac{\partial u(x,t)}{\partial x}\right)_x + 3\left(u(x,t)\dfrac{\partial u](x,t)}{\partial x}\right) \\ \qquad +2\left(u(x,t)-u^2(x,t)\right) \\ u(x,0) = 2\sqrt{e^x - e^{-4x}} \end{cases}, \tag{3.31}$$

either

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} = &\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right)_x + 3\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right) \\ &+ 2\left(u(x,t)-u^2(x,t)\right). \end{aligned} \tag{3.32}$$

Assuming that the non-linear part is

$$\begin{aligned} Nu(x,t) = \frac{\partial u(x,t)}{\partial t} = &\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right)_x \\ &+ 3\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right) - 2u^2(x,t), \end{aligned} \tag{3.33}$$

either

$$\frac{\partial u(x,t)}{\partial t} = Nu(x,t) + 2u(x,t). \tag{3.34}$$

Applying the Laplace transform to the relationship (3.34) gives

$$\mathcal{L}\left(\frac{\partial u(x,t)}{\partial t}\right) = \mathcal{L}\left(Nu(x,t)\right) + 2\mathcal{L}(u(x,t)), \tag{3.35}$$

$$s\mathcal{L}(u(x,t)) - 2\mathcal{L}(u(x,t)) = u(x,0) + \mathcal{L}\left(Nu(x,t)\right), \tag{3.36}$$

and after transforming the writing, the result is

$$\mathcal{L}(u(x,t)) = \frac{1}{s-2}u(x,0) + \frac{1}{s-2}\mathcal{L}\left(Nu(x,t)\right). \tag{3.37}$$

Applying the inverse Laplace transform yields the following (3.38) result:

$$u(x,t) = \mathcal{L}^{-1}(\frac{1}{s-2}u(x,0)) + \mathcal{L}^{-1}(\frac{1}{s-2}\mathcal{L}\left(Nu(x,t)\right)). \tag{3.38}$$

Let's look for the solution of $u(x,t)$ in the form (3.39) below:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{3.39}$$

and

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n(x,t). \tag{3.40}$$

The following algorithm is derived from this:

$$\begin{cases} u_0(x,t) = 2e^{2t}\sqrt{e^x - e^{-4x}} \\ u_{n+1}(x,t) = \mathcal{L}^{-1}(\frac{1}{s-2}\mathcal{L}(A_n((x,t)) \end{cases}, \tag{3.41}$$

$$\begin{cases} u_0(x,t) = 2e^{2t}\sqrt{e^x - e^{-4x}} \\ u_{n+1}(x,t) = \mathcal{L}^{-1}(\frac{1}{s-2}\mathcal{L}(A_n((x,t)) \end{cases}, \tag{3.42}$$

with:

$$A_0(x,t) = \left(u_0(x,t)\frac{\partial u_0(x,t)}{\partial x}\right)_x + 3\left(u_0(x,t)\frac{\partial u_0(x,t)}{\partial x}\right) - 2u_0^2(x,t), \tag{3.43}$$

and

$$\begin{aligned} A_1(x,t) &= \left(u_0\frac{\partial u_1(x,t)}{\partial x} + u_1(x,t)\frac{\partial u_0(x,t)}{\partial x}\right)_x \\ &\quad + 3\left(u_0(x,t)\frac{\partial u_1(x,t)}{\partial x} + u_1\frac{\partial u_0(x,t)}{\partial x}\right) - 4u_0u_1(x,t). \end{aligned} \tag{3.44}$$

The calculations give

(3.45)
$$\begin{cases} A_0(x,t)=0, & u_1(x,t)=0 \\ A_1(x,t)=0, & u_2(x,t)= \\ A_2(x,t)=0, & u_3(x,t)=0 \\ \cdots & \\ u_n(x,t)=0. & \end{cases}.$$

The exact solution to the problem is:

(3.46)
$$u(x,t)=2e^{2t}\sqrt{e^x-e^{-4x}}.$$

4. Application of the variational iteration method

4.1. **Problem 3.** Consider the following problem [6] that has been approached using the Laplace-Adomian method as problem 2:

(4.1)
$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t}=\dfrac{\partial^2 u(x,t)}{\partial x^2}-bu^3(x,t) \\ u(x,0)=\sqrt{\dfrac{2}{b}}\left(\dfrac{2x}{x^2+1}\right) \end{cases},$$

either

(4.2)
$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}-bu^3(x,t).$$

The variational iteration correction functional of the relationship (4.2) yields

(4.3)
$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(t)\left[\frac{\partial u_n}{\partial t}-\frac{\partial^2 u_n(x,s)}{\partial x^2}+bu_n^3(x,s)\right]ds.$$

From the functional (4.3), it follows

(4.4)
$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\int_0^t \lambda(s)\delta\frac{\partial u_n(x,s)}{\partial s}ds=0.$$

Integration by parts of the relationship (4.4) gives:

(4.5)
$$\delta u_{n+1}(x,t)=(1+\delta(t))u_n(x,t)-\int_0^t \lambda'(s)\delta u_n(x,s)ds=0.$$

This leads to stationary conditions:

$$\begin{cases} 1+\lambda(t)=0 \Rightarrow \lambda(t)=-1 \\ \lambda'(s)=0 \end{cases}, \tag{4.6}$$

either: $\lambda(t)=-1$. Consequently, the shape of the variational iterations can be obtained according to the following process:

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left[\frac{\partial u_n}{\partial t}-\frac{\partial^2 u_n(x,s)}{\partial x^2}+bu_n^3(x,s)\right]ds. \tag{4.7}$$

By taking

$$u_0(x,t)=u(x,0)=\sqrt{\frac{2}{b}}\left(\frac{2x}{x^2+1}\right), \tag{4.8}$$

the following approximations are generated

$$u_1(x,t)=u_0(x,t)-\int_0^t\left[\frac{\partial u_0}{\partial t}-\frac{\partial^2 u_0(x,s)}{\partial x^2}+bu_0^3(x,s)\right]ds, \tag{4.9}$$

$$u_1(x,t)=\left(1-\frac{6t}{x^2+1}\right)u_0(x,t), \tag{4.10}$$

$$u_2(x,t)=u_1(x,t)-\int_0^t\left[\frac{\partial u_1}{\partial t}-\frac{\partial^2 u_0(x,s)}{\partial x^2}+bu_1^3(x,s)\right]ds, \tag{4.11}$$

$$u_2(x,t)=\left(1-\frac{6t}{x^2+1}+\frac{36t^2}{(x^2+1)^2}\right)u_0(x,t), \tag{4.12}$$

$$u_3(x,t)=u_2(x,t)-\int_0^t\left[\frac{\partial u_2}{\partial t}-\frac{\partial^2 u_2(x,s)}{\partial x^2}+bu_2^3(x,s)\right]ds, \tag{4.13}$$

$$u_3(x,t)=\left(1-\frac{6t}{x^2+1}+\frac{36t^2}{(x^2+1)^2}-\frac{216t^3}{(x^2+1)^3}\right)u_0(x,t). \tag{4.14}$$

Step by step, the result is

$$u_n(x,t)=\left(1-\frac{6t}{x^2+1}+\frac{36t^2}{(x^2+1)^2}-\frac{216t^3}{(x^2+1)^3}+...+\frac{(-6t)^n}{(x^2+1)^n}\right)u_0(x,t), \tag{4.15}$$

$$u_n(x,t) = u_0(x,t)\left[\frac{1-\left(\frac{-6t}{x^2+1}\right)^N}{x^2+6t+1}\right]. \tag{4.16}$$

Here,

$$u(x,t) = \lim_{N\to+\infty} u_n(x,t). \tag{4.17}$$

Hence the exact solution:

$$u(x,t) = \sqrt{\frac{2}{b}}\left(\frac{2x}{x^2+6t+1}\right). \tag{4.18}$$

5. Problem 4

$$\left\{\begin{array}{l} \dfrac{\partial u(x,t)}{\partial t} = \left(u(x,t)\dfrac{\partial u(x,t)}{\partial x}\right)_x + 3\left(u(x,t)\dfrac{\partial u(x,t)}{\partial x}\right) \\ \qquad +2\left(u(x,t)-u^2(x,t)\right) \\ u(x,0) = 2\sqrt{e^x - e^{-4x}} \end{array}\right., \tag{5.1}$$

either:

$$\frac{\partial u(x,t)}{\partial t} = \left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right)_x + 3\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right) + 2\left(u(x,t)-u^2(x,t)\right). \tag{5.2}$$

The correction functional according to the variational iteration method of the relationship (5.2) gives:

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t)\Bigg[\frac{\partial u_n}{\partial t} - \left(u_n\frac{\partial u_n}{\partial x}\right)_x \\ -3\left(u_n\frac{\partial u_n)}{\partial x}\right) - 2\left(u_n - u_n^2\right)\Bigg] ds. \end{aligned} \tag{5.3}$$

From the relationship (5.3), we have:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \lambda(s)\delta\frac{\partial u_n(x,s)}{\partial s} ds = 0. \tag{5.4}$$

Integration by parts of the equation (5.4), gives:

$$\delta u_{n+1}(x,t) = (1+\delta(t))u_n(x,t) - \int_0^t \lambda^{'}(s)\delta u_n(x,s)ds = 0. \tag{5.5}$$

The above result leads to the following stationary conditions:

$$\begin{cases} 1+\lambda(t)=0 \Rightarrow \lambda(t)=-1 \\ \lambda'(s)=0 \end{cases}, \tag{5.6}$$

which gives $\lambda(t)=-1$.

Consequently, the shape of the variational iterations can be obtained using the following process:

$$\begin{aligned} u_{n+1}(x,t)=u_n(x,t)-\int_0^t \Bigg[\frac{\partial u_n}{\partial t}-\left(u_n\frac{\partial u_n}{\partial x}\right)_x-3\left(u_n\frac{\partial u_n}{\partial x}\right) \\ -2\left(u_n-u_n^2\right)\Bigg]\,ds. \end{aligned} \tag{5.7}$$

By taking $u_0(x,t)=u(x,0)=2\sqrt{e^x-e^{-4x}}$, the following approximations follow:

$$u_1(x,t)=u_0(x,t)-\int_0^t \left[\frac{\partial u_0}{\partial t}-\left(u_0\frac{\partial u_0}{\partial x}\right)_x-3\left(u_0\frac{\partial u_0}{\partial x}\right)-2\left(u_0-u_0^2\right)\right]ds, \tag{5.8}$$

$$u_1(x,t)=(1+2t)(2\sqrt{e^x-e^{-4x}}), \tag{5.9}$$

$$\begin{aligned} u_2(x,t)=u_1(x,t)-\int_0^t \Bigg[\frac{\partial u_1}{\partial t}-\left(u_1\frac{\partial u_1}{\partial x}\right)_x-3\left(u_1\frac{\partial u_1)}{\partial x}\right) \\ -2\left(u_1-u_1^2\right)\Bigg]\,ds, \end{aligned} \tag{5.10}$$

$$u_2(x,t)=(1+2t+2t^2)(2\sqrt{e^x-e^{-4x}}), \tag{5.11}$$

$$\begin{aligned} u_3(x,t)=u_2(x,t)-\int_0^t \Bigg[\frac{\partial u_2}{\partial t}-\left(u_2\frac{\partial u_2}{\partial x}\right)_x-3\left(u_2\frac{\partial u_2)}{\partial x}\right) \\ -2\left(u_2-u_2^2\right)\Bigg]\,ds, \end{aligned} \tag{5.12}$$

$$u_3(x,t)=(1+2t+2t^2+\frac{4}{3}t^3)(2\sqrt{e^x-e^{-4x}}), \tag{5.13}$$

$$\begin{aligned} u_4(x,t)=u_3(x,t)-\int_0^t \Bigg[\frac{\partial u_3}{\partial t}-\left(u_3\frac{\partial u_3}{\partial x}\right)_x-3\left(u_3\frac{\partial u_3)}{\partial x}\right) \\ -2\left(u_3-u_3^2\right)\Bigg]\,ds, \end{aligned} \tag{5.14}$$

$$u_4(x,t)=(1+2t+2t^2+\frac{4}{3}t^3+\frac{2}{3}t^4)(2\sqrt{e^x-e^{-4x}}). \tag{5.15}$$

One step at a time, the result is

$$u_n(x,t) = \left(1 + 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4 + ... + \frac{(2t)^n}{n!})(2\sqrt{e^x - e^{-4x}}\right), \tag{5.16}$$

either:

$$\begin{aligned} u_n(x,t) = &\left(1 + (2t) + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \cdots + \frac{(2t)^n}{n!}\right) \\ &\cdot \left(2\sqrt{e^x - e^{-4x}}\right), \end{aligned} \tag{5.17}$$

$u(x,t) = \lim_{n\to+\infty} u_n(x,t)$. Hence the exact solution:

$$u(x,t) = 2e^{2t}\sqrt{e^x - e^{-4x}}. \tag{5.18}$$

6. Coclusion

In this work, we used Laplace-Adomian methods and variational iterations to solve certain types of partial differential equations. These methods enabled us to obtain exact solutions. We have demonstrated the effectiveness of the Laplace technique, combined with the Adomian method, for solving nonlinear partial differential equations. In addition, the iterative terms of the variational iteration method have led us to the exact solutions of these equations. Both methods are effective for these types of problem

References

[1] S. Ali, L. Humaira, and K. Shah: *Analytical solution of general Fisher's equation by using Laplace Adomian decomposition method*, Pure Appl. Math. **2**(3) (2018), 1–4.

[2] S.N. Ameina: *Power Series Solutions for Nonlinear Systems of Partial Differential Equations*, Applied Mathematical Sciences. **6**(104) (2012), 5147–5159.

[3] S. Duangpithak: *Variational iteration method for special nonlinear partial differential equations*, Int. Journal of Math. Analysis. **6**(22) (2012), 1071–1077.

[4] E.M. DE Jager, J.F Furu: *The theory of singular perturbations*, Elsevier, 1996.

[5] J.S Ghorbaniand, J. S. Nadjafi: *Convergence of He's variational iteration method for nonlinear oscillators*, Nonlinear Sci, **1**(4) (2010), 379–384.

[6] S. Gupta, M. Goyal, A. Prakash: *Numerical Treatment of Newell-Whitehead-Segel Equation*, TWMS. J. App. Eng. Math. **10**(2) (2020), 312–320.

[7] J.H. He: *Variational iteration method a kind of non linear analytical technique: some exemples*, Int. J.Nonlinear Mech. **34** (1999), 699–708.

[8] A.K. HUSSAIN, F.S. FADHEL, Z.R. YAHYA, N. RUSLI: *Variational Iteration Method (VIM) for solving partial integro-differential equations*, Journal of Theoretical & Applied Information Technology. **88**(2) (2016), 367–374.

[9] H.KHAN, R. SHAH, P. KUMAN, D. BALEANU, M. ARIF: *Laplace decomposition for solving nonlinear system of fractional order partial differential equations*, Advances in Difference Equations, SpringerOpen. **2020**(1) (2020), 1–18.

[10] O. KIYMAZ: *An Algorithm for Solving Initial Value Problems Using Laplace-Adomian Decomposition Method*, CApplied Mathematical Sciences. **3**(30) (2009), 1453–1459.

[11] M.Z. MOHAMED, T.M. ELZAKI: *Comparison between the Laplace decomposition method and Adomian decomposition in time-space fractional nonlinear fractional differential equations*, Applied Mathematics, Scientific Research Publishing. **9**(4) (2018), 448–458.

[12] S. MAHMOOD, R. SHAN, M. ARIF: *Laplace Adomian Decomposition Method for Multi Dimensional Time Fractional Model of Navier-Stokes Equation*, Symmetry MDPI. **11**(149) (2019), 1–15.

[13] A. NAGHIPOUR, J. MANAFIAN: *Application of the Laplace Adomian Decomposition and implicit Methods for solving Burger's Equation*, TWMS J. Appl. Math. **6**(1) (2015), 68–77.

[14] Z.M. ODIBAT: *A Study on the Convergence of Variational iterational method*, Mathematical and Computer Modelling,. **39**(5) (1987), 1181–1192.

[15] S.A OJOBOR, A. OBIHIA: *Modified variational iteration method for solving nonlinear partial differential equation using Adomian polynomials*, Mathematics and Statistics. **9**(4) (2021), 456–464.

[16] P. PUE-ON: *Laplace Adomian Decomposition Method for Solving Newell-Whitehead-Segel Equation*, Applied Mathematical Sciences. **7** (2013), 6593–6600.

[17] P. SINGH, D. SHARMA: *Comparative study of homotopy perturbation transformation with homotopy perturbation Elzaki transform method for solving nonlinear fractional PDE*, Nonlinear Engineering, De Gruyter. **9**(1) (2019), 60–71.

[18] A.M. WAZWAZ, M.S. MEHANNA: *The Combined Laplace-Adomian Method for handling Singular Integral Equation of Heat Transfer*, International Journal of Nonlinear Sciences. **10**(2) (2010), 248–252.

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