J COMP SCI APPL MATH
Journal of Computer Science and Applied Mathematics
Vol. 7, no.1, (2025), 1–10
ISSN: 1857-9582 https://doi.org/10.37418/jcsam.7.1.1

ON THE APPROXIMATION BY URYSOHN-TYPE NONLINEAR INTEGRAL OPERATORS

Gümrah Uysal[1], Pelin Söğüt, and Sevgi Esen Almali

ABSTRACT. In this paper, we prove some approximation theorems for Urysohn-type nonlinear integral operators at $\mu-$Lebesgue points of integrable functions. We carry out this examination in two directions such that integration domain being finite and infinite.

1. INTRODUCTION

For each real parameter $\omega \in \Lambda,$ linear integral operators in the following unified form:

$$\mathbf{L}_{\omega}(f;s)=\int_{\mathcal{D}} f(t)\,\mathcal{K}_{\omega}(t,s)\,dt,\quad s\in\mathcal{D},$$

are widely studied throughout years. Here, $\mathcal{K}_{\omega}(t,s)$ is a kernel function, $\mathcal{D}$ is an integration domain and Λ is a non-empty index set. Some prominent studies on this subject can be given as [9, 10, 18, 20]. The problem of approximation by nonlinear integral operators has been important for many years. The solution to this problem was given by Musielak [16] via imposing a strong Lipschitz condition on the kernel function. Some studies that followed this study can

[1]*corresponding author*
Key words and phrases. Urysohn-Type Integral Operators, Pointwise Convergence, Analytic Function.
Submitted: 23.12.2024; *Accepted:* 08.01.2025; *Published:* 16.01.2025.

be given as [7, 8, 17, 19]. In addition to these studies, approximation results were also obtained with integral operators involving power nonlinearity using different methods. [2, 13] can be given as examples of such studies. In some studies, the relevant kernel was chosen everywhere analytic to eliminate the nonlinearity of the kernel (see, e.g., [1, 12]).

The family of Urysohn-type nonlinear integral operators

$$\mathcal{L}_{\omega}(v;s)=\int_{\mathcal{R}}\mathcal{K}_{\omega}(s,t,v(t))\,dt,\ s\in\mathcal{R}, \tag{1.1}$$

where $\mathcal{K}_{\omega}:\mathcal{R}\times\mathcal{R}\times\mathcal{R}\to\mathcal{R}$, ω is a positive real parameter with $\omega\to+\infty$ and $\mathcal{R}$ stands for the set of all real numbers (see [15, 21]). Gadjiev [11] investigated the convergence of Hammerstein-type integrals by setting $\mathcal{K}_{\omega}(s,t,v(t))=\mathcal{H}_{\omega}(s,t)\,\mathcal{G}(t,v(t))$. Almali [3] studied Urysohn-type nonlinear integral operators of type (1.1). In this work, the convergence at Lebesgue points of integrable functions was studied. In the year 2022, Almalı and Kayabaşı [4] generalized this study by obtaining convergence results for $\mathcal{L}_{\omega}(v;s)$ at $p-$Lebesgue points of integrable functions. Some other related studies may be given as [5, 6].

The current study is a generalization of [3]. We first prove an approximation theorem for Urysohn-type nonlinear integral operators of type (1.1) at $\mu-$Lebesgue points of integrable functions. $\mu-$Lebesgue points are the natural generalizations of the Lebesgue points with respect to the function $\mu(t)$ whose characterization was given in [10]. In the current work, for formal definition whenever we mention $\mu-$Lebesgue point, we refer to [14]. Then, we restrict the domain of integration to arbitrary bounded interval $(a,b)\subset\mathcal{R}$ and prove a second theorem for this case. Since the pointwise convergence is considered on a set (Fatou-type convergence), our presented theorems are stated in the form of [10, 14, 18, 20].

2. POINTWISE APPROXIMATION

Let $v:\mathcal{R}\to\mathcal{R}$ and $v_0:=v(s_0)$, where $s_0\in\mathcal{R}$. For any fixed real number s_0, the following properties, which are imposed on the (kernel) function $\mathcal{K}_{\omega}$, are quoted from [3]:

i: The function $\mathcal{K}_\omega(s,t,v)$ is an everywhere analytic function with respect to variable v for every $s,t \in \mathcal{R}$ and $\omega > 0$.

ii: $\lim_{\omega\to+\infty} \int_{\mathcal{R}} \mathcal{K}_\omega(s,t,v_0)\,dt = v_0$ for every $s \in \mathcal{R}$.

iii: $\mathcal{K}_\omega^{(n)}(s_0,t,v_0)$ is monotonically increasing for $t < s_0$ and monotonically decreasing for $t > s_0$ as a function of t.

iv: For every $s,z \in \mathcal{R}$, $\mathcal{K}_\omega^{(n)}(s,z,v_0) \geq 0$ and $\mathcal{K}_\omega^{(n)}(s,z,v_0) \leq \alpha(\omega)$ holds for $n = 1,2,...$, where $\alpha(\omega) \to 0$ as $\omega \to +\infty$.

v: For every $n = 1,2,...$ and $z \neq s_0$, $\mathcal{K}_\omega^{(n)}(s_0,z,v_0) \leq \mathcal{K}_\omega^{'}(s_0,z,v_0)$ holds.

vi: For every $n = 1,2,...$, $\int_{\mathcal{R}} \mathcal{K}_\omega^{(n)}(s_0,t,v_0)\,dt = A_n$, where the numbers A_n may depend only on s_0.

vii: For every $z \neq s_0$, $\lim_{\omega\to+\infty} \mathcal{K}_\omega^{'}(s_0,z,v_0) = 0$.

viii: For every $\xi > 0$, $\lim_{\omega\to+\infty} \int_{|t-s_0|\geq\xi} \mathcal{K}_\omega^{'}(s_0,t,v_0)\,dt = 0$.

Now, we prove a theorem on the convergence of the family of Urysohn-type nonlinear integral operators at $\mu-$Lebesgue point of $v \in L_1(\mathcal{R})$.

Theorem 2.1. *Let the kernel $\mathcal{K}_\omega(s,t,v)$ satisfy conditions (i)-(viii). Then, at each $\mu-$Lebesgue point $s_0 \in \mathcal{R}$ of the function $v \in L_1(\mathcal{R})$, which is bounded on $\mathcal{R}$, there holds that*

$$\lim_{\omega\to+\infty} \mathcal{L}_\omega(v;s_0) = v(s_0)$$

on any set Ω_1 on which the function

$$\int_{s_0-\delta}^{s_0+\delta} \left| \{\mu(|s_0-t|)\}_t^{'} \right| \mathcal{K}_\omega^{'}(s_0,t,v_0)\,dt, \quad 0 < \delta < \delta_1$$

is bounded as $\omega \to +\infty$. Here, δ_1 is a sufficiently large real number.

Proof. We mainly follow the proof steps of [3] with some additional considerations. In view of condition (i), Taylor expansion of $\mathcal{K}_\omega$ at $v = v(s_0)$ can be stated as

$$\mathcal{K}_\omega(s,t,v(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{K}_\omega^{(n)}(s,t,v_0)\,[v(t) - v(s_0)]^n.$$

Under the assumptions, we get

$$\begin{aligned}\left|\mathcal{L}_{\omega}\left(v;s_{0}\right)-v\left(s_{0}\right)\right| &\leq \sum_{n=1}^{\infty}\frac{1}{n!}\int\limits_{\mathcal{R}}\left|v\left(t\right)-v\left(s_{0}\right)\right|^{n}\mathcal{K}_{\omega}^{(n)}\left(s_{0},t,v_{0}\right)dt\\ &+\left|\int\limits_{\mathcal{R}}\mathcal{K}_{\omega}\left(s_{0},t,v_{0}\right)dt-v\left(s_{0}\right)\right|.\end{aligned}$$

Suppose $v\neq 0$ on $\mathcal{R}$. Since v is bounded on $\mathcal{R}$, there exists $B>0$ for every $t\in\mathcal{R}$ such that $\left|v\left(t\right)\right|\leq B$. So, for $n=1,2,...$, we have

$$\left|v\left(t\right)-v\left(s_{0}\right)\right|^{n}\leq\left(2B\right)^{n-1}\left|v\left(t\right)-v\left(s_{0}\right)\right|. \tag{2.1}$$

We know that s_0 is a $\mu-$Lebesgue point of the function $v\in L_{1}\left(\mathcal{R}\right)$, therefore for every $\varepsilon>0$, there exists $\delta>0$ such that $0<h\leq\delta$ and in view of (2.1), we have the following inequalities:

$$\int\limits_{s_{0}-h}^{s_{0}}\left|v\left(t\right)-v\left(s_{0}\right)\right|^{n}dt<\left(2B\right)^{(n-1)}\varepsilon\mu\left(h\right) \tag{2.2}$$

and

$$\int\limits_{s_{0}}^{s_{0}+h}\left|v\left(t\right)-v\left(s_{0}\right)\right|^{n}dt<\left(2B\right)^{(n-1)}\varepsilon\mu\left(h\right). \tag{2.3}$$

Here, the function $\mu:\mathcal{R}\to\mathcal{R}$ is an increasing and absolutely continuous function on $[0,\delta_1]$ with $\mu(0)=0$ (see [10, 14]).

For above mentioned $\delta>0$, we split the integral as follows:

$$\begin{aligned}\left|\mathcal{L}_{\omega}\left(v;s_{0}\right)-v\left(s_{0}\right)\right| &\leq \sum_{n=1}^{\infty}\frac{1}{n!}\int\limits_{|t-s_{0}|\leq\delta}\left|v\left(t\right)-v\left(s_{0}\right)\right|^{n}\mathcal{K}_{\omega}^{(n)}\left(s_{0},t,v_{0}\right)dt\\ &+\sum_{n=1}^{\infty}\frac{1}{n!}\int\limits_{|t-s_{0}|\geq\delta}\left|v\left(t\right)-v\left(s_{0}\right)\right|^{n}\mathcal{K}_{\omega}^{(n)}\left(s_{0},t,v_{0}\right)dt\\ &+\left|\int\limits_{\mathcal{R}}\mathcal{K}_{\omega}\left(s_{0},t,v_{0}\right)dt-v\left(s_{0}\right)\right| =:\sum_{n=1}^{\infty}\frac{1}{n!}\left(\mathbf{I}_{1}+\mathbf{I}_{2}\right)+\mathbf{I}_{3}.\end{aligned}$$

We first examine the integral $\mathbf{I}_2$. In view of condition (v), we have

$$\mathbf{I}_2 \leq \int\limits_{|t-s_0|\geq\delta} (2B)^{(n-1)} |v(t) - v(s_0)| \mathcal{K}_\omega^{'}(s_0, t, v_0)\, dt$$

and

$$\mathbf{I}_2 \leq (2B)^{(n-1)} \left[\int\limits_{|t-s_0|\geq\delta} |v(t)| \mathcal{K}_\omega^{'}(s_0, t, v_0)\, dt + |v(s_0)| \int\limits_{|t-s_0|\geq\delta} \mathcal{K}_\omega^{'}(s_0, t, v_0)\, dt \right].$$

In view of condition (iii), we obtain

$$\begin{aligned} \mathbf{I}_2 &\leq (2B)^{(n-1)} \|v\|_{L_1(\mathcal{R})} \left[\mathcal{K}_\omega^{'}(s_0, s_0 - \delta, v_0) + \mathcal{K}_\omega^{'}(s_0, s_0 + \delta, v_0) \right] \\ &+ (2B)^{(n-1)} |v(s_0)| \int\limits_{|t-s_0|\geq\delta} \mathcal{K}_\omega^{'}(s_0, t, v_0)\, dt. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n!} \mathbf{I}_2 &\leq \sum_{n=1}^{\infty} \frac{1}{n!} (2B)^{(n-1)} \|v\|_{L_1(\mathcal{R})} \left[\mathcal{K}_\omega^{'}(s_0, s_0 - \delta, v_0) + \mathcal{K}_\omega^{'}(s_0, s_0 + \delta, v_0) \right] \\ &+ \sum_{n=1}^{\infty} \frac{1}{n!} (2B)^{(n-1)} |v(s_0)| \int\limits_{|t-s_0|\geq\delta} \mathcal{K}_\omega^{'}(s_0, t, v_0)\, dt. \end{aligned}$$

Under the conditions (vii) and (viii), $\sum_{n=1}^{\infty} \frac{1}{n!} \mathbf{I}_2 \to 0$ as $\omega \to +\infty$. Also, by condition (ii), $\mathbf{I}_3 \to 0$ as $\omega \to +\infty$.

Now, we consider $\mathbf{I}_1$. We write

$$\begin{aligned} \mathbf{I}_1 &= \int\limits_{s_0-\delta}^{s_0} |v(t) - v(s_0)|^n \mathcal{K}_\omega^{(n)}(s_0, t, v_0)\, dt + \int\limits_{s_0}^{s_0+\delta} |v(t) - v(s_0)|^n \mathcal{K}_\omega^{(n)}(s_0, t, v_0)\, dt \\ &=: \mathbf{I}_{11} + \mathbf{I}_{12}. \end{aligned}$$

We now evaluate $\mathbf{I}_{11}$. To do this, we define auxiliary function $G(t)$ as

$$G(t) := \int\limits_{t}^{s_0} |v(u) - v(s_0)|^n\, du.$$

Then, in view of (2.2), the inequality

$$|G(t)| \leq (2B)^{(n-1)} \varepsilon \mu (s_0 - t) \tag{2.4}$$

holds. Applying (2.4) and two times integration by parts method to $\mathbf{I}_{11}$, we get

$$|\mathbf{I}_{11}| \leq (2B)^{(n-1)} \varepsilon \int\limits_{s_0-\delta}^{s_0} \left| \{\mu (t-s_0)\}_t' \right| \mathcal{K}_\omega' (s_0, t, v_0)\, dt. \tag{2.5}$$

Making similar operations as in $\mathbf{I}_{11}$, we obtain the following inequality for $\mathbf{I}_{12}$

$$|\mathbf{I}_{12}| \leq (2B)^{(n-1)} \varepsilon \int\limits_{s_0}^{s_0+\delta} \left| \{\mu (s_0-t)\}_t' \right| \mathcal{K}_\omega' (s_0, t, v_0)\, dt. \tag{2.6}$$

Combining (2.5) and (2.6), we get

$$|\mathbf{I}_{11}| + |\mathbf{I}_{12}| \leq (2B)^{(n-1)} \varepsilon \int\limits_{s_0-\delta}^{s_0+\delta} \left| \{\mu (|s_0-t|)\}_t' \right| \mathcal{K}_\omega' (s_0, t, v_0)\, dt.$$

Under the hypothesis, we see that $\sum_{n=1}^{\infty} \frac{1}{n!}\, \mathbf{I}_1 \to 0$ as $\omega \to +\infty$. Thus, the proof is completed. □

Now, we consider the following Urysohn-type integral operators:

$$\mathcal{T}_\omega (v; s) = \int\limits_a^b \mathcal{K}_\omega (s, t, v(t))\, dt, \ \ s \in (a, b),$$

where $\mathcal{K}_\omega : \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$, (a, b) is an arbitrary bounded interval in $\mathcal{R}$ and ω is a positive real parameter with $\omega \to +\infty$.

Theorem 2.2. *Let the kernel $\mathcal{K}_\omega (s, t, v)$ satisfy conditions (i)-(vii). Then, at each $\mu-$Lebesgue point $s_0 \in (a, b)$ of the function $v \in L_1 (a, b)$ with $v : \mathcal{R} \to \mathcal{R}$ which is bounded on (a, b), there holds that*

$$\lim_{\omega \to +\infty} \mathcal{T}_\omega (v; s_0) = v(s_0)$$

on any set Ω_2 on which the function

$$\int\limits_{s_0-\delta}^{s_0+\delta} \left| \{\mu (|s_0-t|)\}_t' \right| \mathcal{K}_\omega' (s_0, t, v_0)\, dt, \ \ 0 < \delta < \delta_2$$

is bounded as $\omega \to +\infty$. Here, δ_2 is a sufficiently large real number such that $(s_0-\delta, s_0+\delta) \subseteq (a, b)$.

Proof. We define the extension function g by

$$g(t) := \begin{cases} v(t), & t \in (a,b), \\ 0, & t \in \mathcal{R} \backslash (a,b). \end{cases}$$

In view of condition (i), Taylor expansion of $\mathcal{K}_\omega$ at $v = v(s_0)$ can be stated as

$$\mathcal{K}_\omega (s,t,v(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{K}_\omega^{(n)} (s,t,v_0) \left[v(t) - v(s_0) \right]^n .$$

We write

$$\begin{aligned} \mathcal{T}_\omega (v;s_0) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_a^b \left[v(t) - v(s_0) \right]^n \mathcal{K}_\omega^{(n)} (s_0,t,v_0)\, dt \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \left[g(t) - v(s_0) \right]^n \mathcal{K}_\omega^{(n)} (s_0,t,v_0)\, dt. \end{aligned}$$

Under the assumptions, we get

$$\begin{aligned} \left| \mathcal{T}_\omega (v;s_0) - v(s_0) \right| &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \int_a^b \left| v(t) - v(s_0) \right|^n \mathcal{K}_\omega^{(n)} (s_0,t,v_0)\, dt \\ &+ \left| \int_{\mathcal{R}} \mathcal{K}_\omega (s_0,t,v_0)\, dt - v(s_0) \right| . \end{aligned}$$

Suppose $v \neq 0$ on (a,b). Since v is bounded on (a,b), there exists $D > 0$ for every $t \in (a,b)$ such that $|v(t)| \leq D$. So, for $n = 1,2,...,$ we have

$$\left| v(t) - v(s_0) \right|^n \leq (2D)^{n-1} \left| v(t) - v(s_0) \right| . \tag{2.7}$$

We know that s_0 is a $\mu-$Lebesgue point of the function $v \in L_1(a,b),$ therefore for every $\varepsilon > 0,$ there exists $\delta > 0$ such that $0 < h \leq \delta$ and in view of (2.7), we have the following inequalities:

$$\int_{s_0-h}^{s_0} \left| v(t) - v(s_0) \right|^n dt < (2D)^{(n-1)} \varepsilon \mu(h) \tag{2.8}$$

and

$$\int_{s_0}^{s_0+h} |v(t)-v(s_0)|^n \, dt < (2D)^{(n-1)} \varepsilon \mu(h). \tag{2.9}$$

Here, the function $\mu : \mathcal{R} \to \mathcal{R}$ is an increasing and absolutely continuous function on $[0,\delta_2]$ with $\mu(0)=0$.

For above mentioned $\delta>0$, we split the integral as follows:

$$\begin{aligned} |\mathcal{T}_\omega(v;s_0)-v(s_0)| &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(s_0-\delta,s_0+\delta)} |v(t)-v(s_0)|^n \mathcal{K}_\omega^{(n)}(s_0,t,v_0)\,dt \\ &+ \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(a,b)\setminus(s_0-\delta,s_0+\delta)} |v(t)-v(s_0)|^n \mathcal{K}_\omega^{(n)}(s_0,t,v_0)\,dt \\ &+ \left| \int_{\mathcal{R}} \mathcal{K}_\omega(s_0,t,v_0)\,dt - v(s_0) \right| =: \sum_{n=1}^{\infty} \frac{1}{n!} (\mathbf{II}_1+\mathbf{II}_2) + \mathbf{II}_3. \end{aligned}$$

We first evaluate the integral $\mathbf{II}_2$. By condition (v), we have

$$\mathbf{II}_2 \leq \int_{(a,b)\setminus(s_0-\delta,s_0+\delta)} (2D)^{(n-1)} |v(t)-v(s_0)| \mathcal{K}_\omega'(s_0,t,v_0)\,dt$$

and

$$\begin{aligned} \mathbf{II}_2 &\leq (2D)^{(n-1)} \left[\int_{(a,b)\setminus(s_0-\delta,s_0+\delta)} |v(t)| \mathcal{K}_\omega'(s_0,t,v_0)\,dt \right. \\ &+ \left. |v(s_0)| \int_{(a,b)\setminus(s_0-\delta,s_0+\delta)} \mathcal{K}_\omega'(s_0,t,v_0)\,dt \right]. \end{aligned}$$

In view of condition (iii), we obtain

$$\begin{aligned} \mathbf{II}_2 &\leq (2D)^{(n-1)} \|v\|_{L_1(a,b)} \left[\mathcal{K}_\omega'(s_0,s_0-\delta,v_0) + \mathcal{K}_\omega'(s_0,s_0+\delta,v_0) \right] \\ &+ (2D)^{(n-1)} |v(s_0)| (b-a) \left[\mathcal{K}_\omega'(s_0,s_0-\delta,v_0) + \mathcal{K}_\omega'(s_0,s_0+\delta,v_0) \right]. \end{aligned}$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n!}\mathbf{II}_2 &\leq \sum_{n=1}^{\infty}\frac{1}{n!}(2D)^{(n-1)}\|v\|_{L_1(a,b)}\left[\mathcal{K}_\omega'(s_0,s_0-\delta,v_0)+\mathcal{K}_\omega'(s_0,s_0+\delta,v_0)\right]\\ &+ \sum_{n=1}^{\infty}\frac{1}{n!}(2D)^{(n-1)}|v(s_0)|(b-a)\left[\mathcal{K}_\omega'(s_0,s_0-\delta,v_0)\right.\\ &+ \left.\mathcal{K}_\omega'(s_0,s_0+\delta,v_0)\right].\end{aligned}$$

By condition (vii), $\sum_{n=1}^{\infty}\frac{1}{n!}\mathbf{II}_2 \to 0$ as $\omega \to +\infty$. Also, by condition (ii), $\mathbf{II}_3 \to 0$ as $\omega \to +\infty$.

Making similar operations as in previous theorem, we obtain for the integral $\mathbf{II}_1$:

$$|\mathbf{II}_1| \leq (2D)^{(n-1)}\varepsilon\int_{s_0-\delta}^{s_0+\delta}\left|\{\mu(|s_0-t|)\}_t'\right|\mathcal{K}_\omega'(s_0,t,v_0)\,dt$$

in view of (2.8) and (2.9). Under the hypothesis, we arrive at $\sum_{n=1}^{\infty}\frac{1}{n!}\mathbf{II}_1 \to 0$ as $\omega \to +\infty$. Thus, the proof is completed. □

References

[1] S.E. Almali, A.D. Gadjiev: *On approximation properties of certain multidimensional nonlinear integrals*, J. Nonlinear Sci. Appl., **9**(5) (2016), 3090-3097.

[2] S.E. Almali, G. Uysal, V.N. Mishra, O.O. Guller: *On singular integral operators involving power nonlinearity*, Korean J. Math., **25**(4) (2017), 483-494.

[3] S.E. Almali: *On pointwise convergence of the family Urysohn-type integral operators,* Math. Methods Appl. Sci., **42**(16) (2019), 5346-5353.

[4] S.E. Almali, D. Kayabaşi: *Convergence of Urysohn-type nonlinear integral operators at $P-$Lebesgue point*, Advances in Mathematics: Scientific Journal, **11**(4) (2022), 335-352.

[5] H. Anar: *Approximation of the integral funnel of the Urysohn type integral operator,* Appl. Math. Comput., **341**(2019) 277-287.

[6] C. Bardaro, G. Vinti: *Urysohn integral operators with homogeneous kernel: approximation properties in modular spaces,* Comment. Math. (Prace Mat.), **42**(2) (2002), 145-182.

[7] C. Bardaro, J. Musilak, G. Vinti: *Nonlinear Integral Operators and Applications, De Gruyter Series in Nonlinear Analysis and Applications,* **9** (2003), xii+201 pp.

[8] C. Bardaro, G. Vinti, H. Karsli: *Nonlinear integral operators with homogeneous kernels: pointwise approximation theorems,* Appl. Anal., **90**(3-4) (2011), 463-474.

[9] P.L. BUTZER, R.J. NESSEL: *Fourier Analysis and Approximation,* Academic Press, New York and London, 1971.

[10] A.D. GADJIEV: *The order of convergence of singular integrals which depend on two parameters*, Special Problems of Functional Analysis and their Appl. to the Theory of Diff. Eq. and the Theory of Func., Izdat. Akad. Nauk Azerbaijan. SSR., (1968), 40-44.

[11] A.D. GADJIEV: *On the closeness to zero of family of nonlinear integral operators of Hammerstein,* Izvestiya Akad. Nauk Azerbaijan SSR., **2** (1966), 32-34.

[12] O. OZALP GULLER, G. UYSAL: *On certain multidimensional nonlinear integrals,* Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, **69**(2), 1356-1367.

[13] O.O. GULLER: *On pointwise convergence of nonlinear integrals in L_p spaces $(1 < p < +\infty)$,* Palestine Polytechnic University-PPU, **10**(2) (2021), 443-451.

[14] H. KARSLI, E. IBIKLI: *On convergence of convolution type singular integral operators depending on two parameters*, Fasc. Math., **38** (2007), 25-39.

[15] M.A. KRASNOSELSKY: *Topological Methods in the Theory of Nonlinear Integral Equations,* Pergamon Press, New York, **45** (1964), 395 pp.

[16] J. MUSIELAK: *On some approximation problems in modular spaces, In: Constructive Function Theory 1981,* (Proc. Int. Conf., Varna, June 1-5, 1981), Publ. House Bulgarian Acad. Sci., Sofia, 1983, 455-461.

[17] J. MUSIELAK: *Approximation by nonlinear singular integral operators in generalized Orlicz spaces*, Comment. Math. Prace Mat., **31** (1991), 79-88.

[18] B. RYDZEWSKA: *Approximation des fonctions par des integrales singulieres ordinaires*, Fasc. Math., **7** (1973), 71-81.

[19] T. SWIDERSKI, E. WACHNICKI: *Nonlinear singular integrals depending on two parameters,* Comment. Math. Prace Mat., **40** (2000), 181-189.

[20] R. TABERSKI: *Singular integrals depending on two parameters*, Prace Mat., **7** (1962), 173-179.

[21] P. URYSOHN: *Sur une classe d'equations integrales non lineaires*, Mat. Sb., **31**(2) (1923), 236-255.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KARABÜK UNIVERSITY, TÜRKIYE.
Email address: fgumrahuysal@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING AND NATURAL SCIENCES, KIRIKKALE UNIVERSITY, TÜRKIYE.
Email address: plnsgt.46@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING AND NATURAL SCIENCES, KIRIKKALE UNIVERSITY, TÜRKIYE
Email address: sevgi_esen@hotmail.com