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FOUR NEW INTEGRAL THEOREMS INVOLVING STARSHAPED FUNCTIONS

Christophe Chesneau

ABSTRACT. This article presents four new integral theorems relating to starshaped functions. The first theorem establishes a fundamental result concerning the integrability of these functions. The second and third theorems provide univariate integral inequalities, and the fourth introduces a bivariate Hilbert-type integral inequality. Detailed proofs are given for each theorem.

1. INTRODUCTION

Studying integral results under convex-type assumptions is essential for extending the scope of classical inequalities and uncovering new analytical relationships. Such assumptions allow for greater flexibility in function behavior while preserving useful structural properties. For further developments in this area, see [7, 8, 10].

In this article, we show how convex-type assumptions can be effectively employed to derive meaningful results in both univariate and bivariate contexts. In particular, we focus on the notion of starshapedness through the concept of a starshaped function. A formal definition is given below.

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Definition 1.1. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $b > a$, and $f : [a, b] \to [0, +\infty)$ be a function. Then f is said to be starshaped if and only if, for any $\tau \in [0, 1]$ and $x \in [a, b]$, we have*

$$f(\tau x) \leq \tau f(x).$$

We refer the reader to [3, 4] for further background. If we take $a = 0$ and $b = +\infty$, typical examples of functions satisfying the desired properties include $f(x) = x^\gamma$ with $\gamma > 1$, $f(x) = \exp(x) - 1$ and $f(x) = \sqrt{x^2 + 1} - 1$. If we take $a = 0$ and $b = 1$, examples are $f(x) = 1 - \cos((\pi/2)x)$, $f(x) = \tan((\pi/4)x)$ and $f(x) = -\log(1 - x)$ (excluding the value $x = 1$), to name a few.

The starshapedness assumption is one of the most fundamental convex-type assumptions. It is often viewed as a special case of the broader m-convexity framework, specifically corresponding to the case $m = 0$. Within this setting, a wide range of integral inequalities, particularly of the Hermite-Hadamard type and Jensen type, have been established in the literature. See [1,2,5,6,9,11]. Focusing exclusively on the starshapedness assumption, this article presents a collection of simple yet previously unpublished univariate integral results, as well as a new bivariate Hilbert-type integral inequality. These findings are structured into four theorems, each supported by a complete and self-contained proof.

The remainder of the article is organized as follows: Section 2 presents the main theorems along with their detailed proofs. Section 3 offers concluding remarks and perspectives for future research.

2. Contributions

2.1. First theorem.

The result below is an univariate integrability result for starshaped functions. The proof is based on reasoning by contradiction using a change of variables technique.

Theorem 2.1. *Let $f : [0, +\infty) \to [0, +\infty)$ be a starshaped function. Then it is not integrable on $[0, +\infty)$, i.e.,*

$$\int_0^{+\infty} f(x)dx = +\infty.$$

Proof. A contradiction reasoning is used. Let us assume that f is integrable on $[0, +\infty)$ and consider $\tau \in [0, 1)$. Performing the change of variables $x = \tau y$, and using the fact that f is starshaped, i.e., $f(\tau y) \leq \tau f(y)$, we have

$$\begin{aligned}\int_0^{+\infty} f(x)dx &= \int_0^{+\infty} f(\tau y)\tau dy \\ &\leq \int_0^{+\infty} \tau f(y)\tau dy = \tau^2 \int_0^{+\infty} f(y)dy = \tau^2 \int_0^{+\infty} f(x)dx.\end{aligned}$$

We simplify the integral of f on both sides, which is possible because of its supposed convergence, and get

$$\tau^2 \geq 1.$$

This contradicts $\tau \in [0, 1)$. As a result, f cannot be integrable on $[0, +\infty)$. The proof ends. □

This result shows that caution must be exercised when working with integrals involving starshaped functions, as there is a risk of encountering an integrability problem when integrating over the interval $[0, +\infty)$.

2.2. **Second theorem.** The result below is an univariate integral inequality involving the power of a starshaped function. The proof is based on a change of variables technique and standard power integral calculus.

Theorem 2.2. *Let $\alpha > 0$ and $f : [0, \alpha] \to [0, +\infty)$ be a starshaped function. Then, for any $\epsilon \geq 0$, we have*

$$\int_0^{\alpha} f^{\epsilon}(x)dx \leq \frac{\alpha}{\epsilon + 1} f^{\epsilon}(\alpha).$$

Proof. Performing the change of variables $x = \alpha y$, and using the fact that f is starshaped, i.e., $f(\alpha y) \leq y f(\alpha)$ with $y \in [0, 1]$, and standard power integral calculus, we have

$$\int_0^{\alpha} f^{\epsilon}(x)dx = \int_0^1 f^{\epsilon}(\alpha y)\alpha dy \leq \int_0^1 [y f(\alpha)]^{\epsilon} \alpha dy = \alpha f^{\epsilon}(\alpha) \int_0^1 y^{\epsilon} dy = \frac{\alpha}{\epsilon + 1} f^{\epsilon}(\alpha).$$

This concludes the proof. □

In particular, for $\epsilon = 1$, we have

$$\int_0^{\alpha} f(x)dx \leq \frac{\alpha}{2} f(\alpha).$$

This corresponds to [4, Lemma 4]. In a sense, this also completes Theorem 2.1, demonstrating how the truncated integral of f can be bounded from above.

Incorporating the adjustable parameter ϵ adds a new degree of flexibility to this established result.

2.3. **Third theorem.** The result below can be presented as an univariate integral inequality involving a starshaped function. The proof is based on a change of variables technique and the Chasles integral relation.

Theorem 2.3. *Let $\alpha > 0$ and $f : [0, +\infty) \to [0, +\infty)$ be a starshaped function. Then, for any $\tau \in [0, 1)$, we have*

$$\int_0^{\alpha/\tau} f(x)dx \leq \frac{1}{1-\tau^2} \int_\alpha^{\alpha/\tau} f(x)dx$$

Proof. Performing the change of variables $x = \tau y$, and using the fact that f is starshaped, i.e., $f(\tau y) \leq \tau f(y)$ with $\tau \in [0, 1)$, we have

$$\begin{aligned}\int_0^{\alpha} f(x)dx &= \int_0^{\alpha/\tau} f(\tau y)\tau dy \leq \int_0^{\alpha/\tau} \tau f(y)\tau dy = \tau^2 \int_0^{\alpha/\tau} f(y)dy \\ &= \tau^2 \int_0^{\alpha/\tau} f(x)dx.\end{aligned}$$

Using the Chasles integral relation, $\tau \in [0, 1)$ so that $\alpha/\tau > \alpha$, and the inequality above, we get

$$\int_0^{\alpha/\tau} f(x)dx = \int_0^{\alpha} f(x)dx + \int_\alpha^{\alpha/\tau} f(x)dx \leq \tau^2 \int_0^{\alpha/\tau} f(x)dx + \int_\alpha^{\alpha/\tau} f(x)dx.$$

This implies that

$$(1-\tau^2) \int_0^{\alpha/\tau} f(x)dx \leq \int_\alpha^{\alpha/\tau} f(x)dx.$$

Since $\tau \in [0, 1)$, we can divided both sides by $1 - \tau^2 > 0$, and we derive

$$\int_0^{\alpha/\tau} f(x)dx \leq \frac{1}{1-\tau^2} \int_\alpha^{\alpha/\tau} f(x)dx.$$

This concludes the proof. □

Thanks to this result, the following inequalities hold:

$$\int_\alpha^{\alpha/\tau} f(x)dx \leq \int_0^{\alpha/\tau} f(x)dx \leq \frac{1}{1-\tau^2} \int_\alpha^{\alpha/\tau} f(x)dx.$$

The first inequality follows immediately from the non-negativity of f. This gives us control over the second integral depending on α and τ, adding a new result to the theory of integral dealing with starshaped functions.

2.4. **Fourth theorem.** Hilbert-type integral inequalities are well-known results in analysis, with many applications in operator theory. Comprehensive treatments can be found in the books [12, 13].

The result below presents a bivariate Hilbert-type integral inequality involving starshaped functions. The proof is based on multiple change of variables techniques, including the polar change of variables, and standard integral calculus techniques.

Theorem 2.4. *Let $\alpha > 0$ and $\beta > 0$ such that $2\alpha \geq \beta$, and $f, g : [0, \max(\alpha, \beta)] \to [0, +\infty)$ be two starshaped functions. Then we have*

$$\iint_{\{(x,y)\in[0,\alpha]^2;\ x+y\leq\beta\}} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{1}{6}\int_0^\beta f(x)g(x)dx.$$

Proof. Performing the change of variables $x = u^2$ and $y = v^2$, we obtain

$$\begin{aligned}
&\iint_{\{(x,y)\in[0,\alpha]^2;\ x+y\leq\beta\}} \frac{1}{x+y} f(x)g(y)dxdy \\
&= \iint_{\{(u,v)\in[0,\sqrt{\alpha}]^2;\ u^2+v^2\leq\beta\}} \frac{1}{u^2+v^2} f(u^2)g(v^2)(4uvdudv) \\
&= 4\iint_{\{(u,v)\in[0,\sqrt{\alpha}]^2;\ u^2+v^2\leq\beta\}} \frac{uv}{u^2+v^2} f(u^2)g(v^2)dudv. \qquad (2.1)
\end{aligned}$$

Performing the polar change of variables $(u, v) = (\rho\cos(\theta), \rho\sin(\theta))$ having the absolute value of the Jacobian equals to ρ, and taking into account that $u, v \geq 0$ which implies $\rho \geq 0$ and $\theta \in [0, \pi/2]$, and

$$u^2 + v^2 \leq \beta \ \Leftrightarrow\ \rho^2\cos^2(\theta) + \rho^2\sin^2(\theta) \leq \beta \ \Leftrightarrow\ \rho^2 \in [0, \beta] \ \Leftrightarrow\ |\rho| \in [0, \sqrt{\beta}],$$

so that $\rho \in [0, \sqrt{\beta}]$, we obtain

$$\begin{aligned}
&4\iint_{\{(u,v)\in[0,\sqrt{\alpha}]^2;\ u^2+v^2\leq\beta\}} \frac{uv}{u^2+v^2} f(u^2)g(v^2)dudv \\
&= 4\int_0^{\sqrt{\beta}}\int_0^{\pi/2} \frac{\rho\cos(\theta)\rho\sin(\theta)}{\rho^2\cos^2(\theta)+\rho^2\sin^2(\theta)} f(\rho^2\cos^2(\theta))g(\rho^2\sin^2(\theta))(\rho d\theta d\rho) \qquad (2.2)
\end{aligned}$$

$$= 4\int_0^{\sqrt{\beta}}\int_0^{\pi/2}\rho\cos(\theta)\sin(\theta)f(\rho^2\cos^2(\theta))g(\rho^2\sin^2(\theta))d\theta d\rho.$$

Using the facts that f and g are starshaped, since $\tau_1 = \cos^2(\theta) \in [0,1]$ and $\tau_2 = \sin^2(\theta) \in [0,1]$, we have

$$f(\rho^2\cos^2(\theta)) = f(\tau_1\rho^2) \le \tau_1 f(\rho^2) = \cos^2(\theta)f(\rho^2)$$

and

$$g(\rho^2\sin^2(\theta)) = g(\tau_2\rho^2) \le \tau_2 g(\rho^2) = \sin^2(\theta)g(\rho^2).$$

These inequalities give

$$\begin{aligned}
&4\int_0^{\sqrt{\beta}}\int_0^{\pi/2}\rho\cos(\theta)\sin(\theta)f(\rho^2\cos^2(\theta))g(\rho^2\sin^2(\theta))d\theta d\rho\\
&\le 4\int_0^{\sqrt{\beta}}\int_0^{\pi/2}\rho\cos(\theta)\sin(\theta)\cos^2(\theta)f(\rho^2)\sin^2(\theta)g(\rho^2)d\theta d\rho\\
&= 4\int_0^{\sqrt{\beta}}\int_0^{\pi/2}\cos^3(\theta)\sin^3(\theta)\rho f(\rho^2)g(\rho^2)d\theta d\rho\\
&= 2\left[\int_0^{\pi/2}\cos^3(\theta)\sin^3(\theta)d\theta\right]\left[\int_0^{\sqrt{\beta}}f(\rho^2)g(\rho^2)(2\rho d\rho)\right]\\
&= 2\left[\int_0^{\pi/2}\cos(\theta)[1-\sin^2(\theta)]\sin^3(\theta)d\theta\right]\left[\int_0^{\sqrt{\beta}}f(\rho^2)g(\rho^2)(2\rho d\rho)\right]. \qquad (2.3)
\end{aligned}$$

Performing the changes of variables $\eta = \sin(\theta)$ for the first integral, and $\omega = \rho^2$ for the second integral, and using standard power integral calculus, we find that

$$\begin{aligned}
&2\left[\int_0^{\pi/2}\cos(\theta)[1-\sin^2(\theta)]\sin^3(\theta)d\theta\right]\left[\int_0^{\sqrt{\beta}}f(\rho^2)g(\rho^2)(2\rho d\rho)\right]\\
&= 2\left[\int_0^1(1-\eta^2)\eta^3 d\eta\right]\left[\int_0^{\beta}f(\omega)g(\omega)d\omega\right]\\
&= 2\left[\int_0^1\eta^3 d\eta - \int_0^1\eta^5 d\eta\right]\left[\int_0^{\beta}f(\omega)g(\omega)d\omega\right]\\
&= \frac{1}{6}\int_0^{\beta}f(\omega)g(\omega)d\omega = \frac{1}{6}\int_0^{\beta}f(x)g(x)dx. \qquad (2.4)
\end{aligned}$$

Joining Equations (2.1), (2.2), (2.3) and (2.4), we finally obtain

$$\iint_{\{(x,y)\in[0,\alpha]^2;\ x+y\leq\beta\}} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{1}{6}\int_0^\beta f(x)g(x)dx.$$

This concludes the proof. □

In particular, if we take $f = g$, we have

$$\iint_{\{(x,y)\in[0,\alpha]^2;\ x+y\leq\beta\}} \frac{1}{x+y} f(x)f(y)dxdy \leq \frac{1}{6}\int_0^\beta f^2(x)dx.$$

The upper bound thus relies on the L_2 integral norm of f.

More generally, note that the case $\beta \to +\infty$ is not of interest, as the upper bound diverges; the product fg remains starshaped, and its integral is infinite by Theorem 2.1. This highlights the importance of considering a bounded domain of integration.

In the upper bound, note also that the constant factor $1/6$ is independent of π, and that the expression involves the integral of both fg, which contrasts with the structure of the classical Hilbert-type integral inequalities. See [12, 13]. To the best of our knowledge, Theorem 2.4 is a new result in the literature and one of the few bivariate inequalities established under the starshapedness assumption.

3. Conclusion

In conclusion, this article introduces four new integral theorems under the starshapedness assumption. The univariate and bivariate inequalities presented here provide a basis for further generalizations, including extensions to other convex-type assumptions, such as the (α, m)-convexity. Future work may also explore multidimensional analogues, applications to fractional integrals, or refinements under additional regularity conditions.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CAEN-NORMANDIE
UFR DES SCIENCES - CAMPUS 2, CAEN
FRANCE.
Email address: christophe.chesneau@gmail.com