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AN ORIGINAL APPROXIMATION INVOLVING PI, THE EULER-MASCHERONI CONSTANT AND THE CATALAN CONSTANT

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ABSTRACT. In this article, we present a simple numerical approximation connecting three well-known mathematical constants: Pi, the Euler-Mascheroni constant and the Catalan constant. We derive an approximation of Pi as a function of the other two constants. The discussion also covers an integral perspective.

1. INTRODUCTION

1.1. **Statement of the main result.** In mathematics, certain constants appear frequently in various branches, ranging from geometry and analysis to number theory. Three well-known examples are Pi, written π, the Euler-Mascheroni constant, written γ, and the Catalan constant, written G. Each of these constants plays a fundamental role in numerous mathematical expressions and theorems. The main result of this article is to relate these constants through integral and approximation formulas, which are as follows:

$$\int_0^{+\infty} \frac{1}{(\gamma x + G)(1 + x^2)} dx = \frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)} \approx 1.0000748.$$

Key words and phrases. numerical approximation, Pi, Euler-Mascheroni constant, Catalan constant.
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This formula is notable for its relative simplicity. Consisting mainly of ratios of polynomial, power and logarithmic functions (without fractional exponents), it achieves a high degree of precision. It also allows us to derive an original approximation of π as a function of γ and G. This result contributes to our understanding of the properties of mathematical constants and their relationships, highlighting the importance of curiosity and exploration in mathematical research.

1.2. **Structure of article.** The rest of the article is divided into two sections. Section 2 contains some background information and the history of the involved constants, as well as some known approximations. Section 3 presents the results and provides further discussion.

2. Background information and history

2.1. Constant definitions, histories and uses.

2.1.1. *The constant* π. The constant π can be defined as the ratio of the circumference of a circle to its diameter. The following approximation applies: $\pi \approx 3.14159$. We can aptly refer to it as "the queen of constants" to emphasize its central role and elegance in many mathematical applications. Its history dates back to ancient civilizations, with approximations found in Babylonian and Egyptian mathematics. Archimedes famously developed a method of approximating π using inscribed and circumscribed polygons, yielding the following bounds:

$$3.14084 \approx \frac{223}{71} < \pi < \frac{22}{7} \approx 3.14285.$$

In modern mathematics, π appears in many fundamental formulas, including Euler's identity $e^{i\pi} + 1 = 0$, where e is the base of the natural logarithm and i denotes the imaginary unit satisfying $i^2 = -1$, the Leibniz series expressed as follows:

$$\pi = 4 \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}$$

and the Gauss integral given by

$$\pi = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx \right)^2 .$$

The use of π spans geometry, calculus, trigonometry, and physics, emphasizing its importance in describing natural and abstract phenomena. For more information on π and its various formulas and approximations, see [1], [7], [6] and [4].

2.1.2. *The constant γ.* The Euler-Mascheroni constant γ is defined by

$$\gamma = \lim_{n\to+\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right) \approx 0.57721.$$

It was introduced by Leonhard Euler in the 18th century and later named after him and the Italian mathematician Lorenzo Mascheroni. It is a key constant in several areas of mathematics, including number theory, analysis and special functions. In modern mathematics, γ appears in many integral representations, such as

$$\gamma = -\int_{0}^{+\infty} e^{-x}\log(x)dx,$$

$$\gamma = \int_{0}^{+\infty}\left(\frac{1}{e^{x}-1} - \frac{1}{xe^{x}}\right)dx$$

and

$$\gamma = \int_{0}^{+\infty}\left(\frac{1}{1+x} - e^{-x}\right)\frac{1}{x}dx.$$

Despite its ubiquity, it remains unknown whether γ is rational or irrational, posing an enduring challenge to number theory. We refer the interested reader to [1], [7], [6], [8] and [12].

2.1.3. *The constant G.* The Catalan constant G is less well known than π and γ. It is defined by the following infinite series:

$$G = \sum_{n=0}^{+\infty}\frac{(-1)^{n}}{(2n+1)^{2}} \approx 0.91596.$$

It was first introduced by Eugène Charles Catalan in the 19th century. It appears in various areas of mathematics, including number theory, analysis and special functions. Some of its most notable representations are the following integrals:

$$G = \int_{0}^{1}\frac{\arctan(x)}{x}dx,$$

$$G = -\int_0^1 \frac{\log(x)}{1+x^2}dx$$

and

$$G = \int_0^{+\infty} \arctan(e^{-x})dx.$$

In addition, G is associated with special functions and appears in relations involving the Clausen function and polylogarithms. Despite its importance, it is not known whether this constant is algebraic or transcendental, and its exact properties are the subject of ongoing research. Further details can be found in [1], [7], [2], [9], [3] and [10].

2.2. **Known approximations between constants.** Accurate approximations of π are numerous. We highlight

$$\pi \approx \frac{355}{113} \approx 3.1415929203539823$$

and the famous one found by Srinivasa Ramanujan, as follows:

$$\pi \approx \frac{9801}{2206\sqrt{2}} \approx 3.1415927300133056.$$

See [4]. Approximations that connect π and γ are relatively rare. We mention the following elegant one:

$$\frac{\pi}{2e} - \gamma \approx 0.0006480099939279$$

We refer to [11]. To the best of our knowledge, there is no known simple formula that connects π, γ and G directly in a single expression. More precisely, these constants occur in different mathematical contexts, and while they may appear together in certain advanced mathematical relationships (e.g., series, integrals, or special functions), there is no simple formula that connects them.

In this article, we do not provide such a "utopian" (or "highly unexpected") formula. However, as described in the introduction, we present a new approximation formula that combines relative simplicity, based mainly on a ratio of polynomial, power and logarithmic functions (there is no fractional exponent), with an interesting degree of precision. We then derive an original approximation to π as a function of γ and G. Next, we explore various aspects of these approximations, including an integral representation of the main term. These points are covered in more detail in the subsequent sections.

3. Results and discussion

3.1. **Numerical proof.** The result below formalizes our main approximation. It establishes a connection between π, γ and G with high decimal precision.

Corollary 3.1. *We have*

$$\frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2+\gamma^2)} - 1 \approx 0.0000748$$

(*followed by:* $6474835927516214929054016846176011864960923121 5968\ldots$).

Proof. Keeping 7 decimals for convenience, we have

$$\pi \approx 3.1415926, \quad \gamma \approx 0.5772156, \quad G \approx 0.9159655.$$

A basic calculus gives

$$\frac{3.1415926 \times 0.9159655 + 2 \times 0.5772156 \times \log(0.5772156/0.9159655)}{2(0.9159655^2 + 0.5772156^2)} - 1$$
$$\approx 1.0000749 - 1 \approx 0.0000749.$$

The desired result is obtained with more decimals in the approximation of π, γ and G, which completes the numerical proof. □

So we have $[\pi G + 2\gamma \log(\gamma/G)]/[2(G^2+\gamma^2)] \approx 1.00007$, and also, with a bit less precision, we can say that

$$\pi G + 2\gamma \log\left(\frac{\gamma}{G}\right) \approx 2(G^2+\gamma^2).$$

Both quantities are approximately equal to 2.344 (the fourth decimals are slightly different).

Note also that an alternative extended form of the main term in Corollary 3.1 is

$$\frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2+\gamma^2)} - 1 = \frac{1}{G^2+\gamma^2}\left[\frac{\pi}{2}G - \gamma\log(G) + \gamma\log(\gamma)\right] - 1$$
$$= \frac{\pi G}{2(G^2+\gamma^2)} - \frac{\gamma\log(G)}{G^2+\gamma^2} + \frac{\gamma\log(\gamma)}{G^2+\gamma^2} - 1 \quad (\approx 0.0000748).$$

As mentioned in the introduction, the approximation in Corollary 3.1 is notable for its relative simplicity. It consists of rational and logarithmic functions, and no parameters need to be adjusted to achieve a significant degree of precision.

The approximation in Corollary 3.1 originates from the daily research routine of the author, which involves computing numerous integrals with different parameters using various symbolic software tools. Among many attempts, one of these integrals, parameterized G and γ, produced an unexpected approximation result (the integral result in question is discussed in Subsection 3.3). After a thorough literature search, no mention of a similar approximation was found, which motivated sharing this surprising result. In a sense, Corollary 3.1 belongs to the family of results discovered by chance rather than design, emphasizing the importance of curiosity and exploration in mathematical research.

3.2. **Approximation.** Given Corollary 3.1, an exotic approximation of π as a function of γ and G is given below.

Corollary 3.2. *We have*

$$\pi \approx \frac{2}{G}\left[G^2 + \gamma^2 - \gamma \log\left(\frac{\gamma}{G}\right)\right] \approx 3.1414$$

(*followed by:* $0104316351608444092945816154606257755475989402077477\ldots$).

Proof. The proof follows from Corollary 3.1; π can be isolated to give the desired result. □

To the best of our knowledge, this result is one of the few attempts to obtain a simple numerical approximation of π as a function of γ and G. This is purely of mathematical interest. However, this approximation is not very precise, and much more accurate ones exist (although these do not involve known constants). See [4].

3.3. **Integral proof.** As mentioned in the introduction, the main approximation is centred on an integral formula. This formula does not depend on π, and uses a simple linear function depending on γ and G, as outlined below.

Proposition 3.1. *We have*

$$\frac{\pi G + 2\gamma \log(\gamma/G)}{2(G^2+\gamma^2)} = \int_0^{+\infty} \frac{1}{(\gamma x + G)(1+x^2)} dx.$$

Proof. Let us introduce the following function:

$$f(x) = \frac{1}{G^2+\gamma^2}\left\{\gamma \log\left[\frac{\gamma x + G}{\sqrt{1+x^2}}\right] + G\arctan(x)\right\}.$$

Then we have

$$\begin{aligned}
f'(x) &= \frac{1}{G^2+\gamma^2}\left\{\gamma \frac{[\gamma\sqrt{1+x^2} - (\gamma x+G)x/\sqrt{1+x^2}]/(1+x^2)}{(\gamma x+G)/\sqrt{1+x^2}} + G\frac{1}{1+x^2}\right\} \\
&= \frac{1}{G^2+\gamma^2}\left\{\gamma \frac{\gamma(1+x^2) - (\gamma x+G)x}{(\gamma x+G)(1+x^2)} + G\frac{1}{1+x^2}\right\} \\
&= \frac{1}{G^2+\gamma^2}\left\{\gamma \frac{\gamma - Gx}{(\gamma x+G)(1+x^2)} + G\frac{\gamma x+G}{(\gamma x+G)(1+x^2)}\right\} \\
&= \frac{1}{G^2+\gamma^2}\left[\frac{G^2+\gamma^2}{(\gamma x+G)(1+x^2)}\right] \\
&= \frac{1}{(\gamma x+G)(1+x^2)}.
\end{aligned}$$

So we get

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{(\gamma x+G)(1+x^2)}dx = \int_0^{+\infty} f'(x)dx = [f(x)]_{x\to 0}^{x\to+\infty} \\
&= \lim_{x\to+\infty}\left[\frac{1}{G^2+\gamma^2}\left\{\gamma \log\left[\frac{\gamma x + G}{\sqrt{1+x^2}}\right] + G\arctan(x)\right\}\right] \\
&- \lim_{x\to 0}\left[\frac{1}{G^2+\gamma^2}\left\{\gamma \log\left[\frac{\gamma x + G}{\sqrt{1+x^2}}\right] + G\arctan(x)\right\}\right] \\
&= \frac{1}{G^2+\gamma^2}\left[\gamma\log(\gamma) + G\frac{\pi}{2} - \gamma\log(G)\right] \\
&= \frac{\pi G + 2\gamma\log(\gamma/G)}{2(G^2+\gamma^2)}.
\end{aligned}$$

This concludes the proof. □

Taking an integral approach, we see that the resulting term contains the value of π, indicating that the complex expression of γ and G comes from the simple linear term $\gamma x + G$.

Combining Corollary 3.1 and Proposition 3.1, we find that

$$\int_0^{+\infty} \frac{1}{(\gamma x+G)(1+x^2)}dx = \frac{\pi G + 2\gamma\log(\gamma/G)}{2(G^2+\gamma^2)} \approx 1.0000748.$$

Note that, since the value 1 is "almost" reached, the exotic positive function $f(x) = 1/[(\gamma x + G)(1 + x^2)]$ depending on γ and G is "almost" a probability density function. Numerical analysis shows that the values of a and b such that $f(x) = 1/[(ax + b)(1 + x^2)]$ is an exact probability density function form a curve and that the point (γ, G) is extremely close to this curve.

These results are related to the approximation in Corollary 3.1, but do not fully explain its sharpness. Further investigation is required, either by extending our integral approach or by utilizing other technical tools, such as series and geometric techniques.

3.4. **Additional note.** During our research, we also found the following approximation:

$$\frac{\pi\gamma - 2\gamma \log(\gamma/G)}{2(G^2 + \gamma^2)} - 1 \approx 0.000899$$

(followed by: $26791258766160770495345878449858711804736495860604\ldots$).

The main term is similar to that in Corollary 3.1, but with two notable changes to the numerator: a γ instead of a G and a minus sign instead of a plus sign before the logarithmic term. The approximation is also less precise. Nevertheless, the similarity in form is intriguing, and the level of precision remains acceptable. This should therefore be regarded as a supplementary remark that maintains a sense of mathematical mystery.

Lastly, interested readers can find a complementary study of this topic in [5], which also involves the golden ratio constant and $\log(2)$.

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