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A NOTE ON A GENERALIZED OLECH-OPIAL INTEGRAL INEQUALITY

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ABSTRACT. This note presents a new, generalized version of the Olech-Opial integral inequality, which is characterized by the inclusion of an intermediate convex function. Notably, this framework relaxes the standard initial condition on the primary function. Several examples are presented to illustrate and validate the theory.

1. INTRODUCTION

The mathematical literature contains numerous integral inequalities, each addressing a specific objective. One such result is the Olech-Opial integral inequality established in [8]. It provides an elegant bound relating a differentiable function and its derivative in integral form. This is stated formally in the theorem below.

Theorem 1.1. *Let $\alpha > 0$ and $f : [0, \alpha] \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$. Then we have*

$$\int_0^\alpha |f(x)f'(x)|dx \leq \frac{\alpha}{2}\int_0^\alpha [f'(x)]^2 dx.$$

This result has inspired numerous advances in the field of theory of differential and integral equations. See [1–7].

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In this note, we present a new generalization based on a convex approach. This approach is inspired by the proof methodology developed in [6] and [2, Theorem 2.3]. Specifically, we introduce an auxiliary convex function with no initial condition that can be adapted to various mathematical contexts. Our method also relaxes the traditional requirement of $f(0) = 0$, thereby extending the scope of the original result. We provide several examples of different convex functions, including power, exponential, logarithmic and trigonometric functions. These lead to new integral inequalities.

The main contribution is presented in Section 2. Section 3 offers concluding remarks and potential directions for future research.

2. Main result with proof

First of all, we recall that a function $\varphi : [0, +\infty) \to \mathbb{R}$ is said to be *convex* if, for any $x, y \in [0, +\infty)$ and any $\theta \in [0, 1]$, the following inequality holds:

$$\varphi(\theta x + (1 - \theta)y) \leq \theta\varphi(x) + (1 - \theta)\varphi(y).$$

If φ is twice differentiable, this is equivalent to $\varphi''(x) \geq 0$ for all $x \in [0, +\infty)$. An important integral inequality associated with the concept of convexity is the Jensen integral inequality, which will play a key role in our study. Further details can be found in [9].

Based on this mathematical foundation, we state the main theorem below. We emphasize the importance of allowing $f(0)$ to be arbitrary, as well as the role of the introduced convex function φ, including its value at the origin, $\varphi(0)$. The proof and examples of this result are presented immediately after the statement.

Theorem 2.1. *Let $\alpha > 0$, $f : [0, \alpha] \to \mathbb{R}$ be a differentiable function and $\varphi : [0, +\infty) \to \mathbb{R}$ be a twice differentiable convex function. Then we have*

$$\int_0^\alpha \varphi'(|f(x) - f(0)|)|f'(x)|dx \leq \frac{1}{\alpha} \int_0^\alpha \varphi\left(\alpha|f'(x)|\right) dx - \varphi(0).$$

Proof. We adopt the proof methodology developed in [6] and [2, Theorem 2.3], making significant changes where the function φ, $\varphi(0)$ and $f(0)$ are involved. Let us set

$$\Phi(x) = \int_0^x |f'(t)|dt.$$

By the definition of a primitive, we get

$$\Phi'(x) = |f'(x)|. \tag{2.1}$$

Applying the triangle inequality, we obtain

$$|f(x) - f(0)| = \left| \int_0^x f'(t)dt \right| \leq \int_0^x |f'(t)|dt = \Phi(x). \tag{2.2}$$

Since φ is twice differentiable and convex, φ' is non-decreasing. This and Equation (2.2) give

$$\varphi'(|f(x) - f(0)|) \leq \varphi'(\Phi(x)). \tag{2.3}$$

It follows from equations (2.1) and (2.3) that

$$\begin{aligned} \int_0^\alpha \varphi'(|f(x) - f(0)|)|f'(x)|dx &= \int_0^\alpha \varphi'(|f(x) - f(0)|)\Phi'(x)dx \\ &\leq \int_0^\alpha \varphi'(\Phi(x))\Phi'(x)dx. \end{aligned} \tag{2.4}$$

Based on this last integral, using the primitive rule for composition, we have

$$\begin{aligned} \int_0^\alpha \varphi'(\Phi(x))\Phi'(x)dx = [\varphi(\Phi(x))]_{x=0}^{x=\alpha} &= \varphi(\Phi(\alpha)) - \varphi(\Phi(0)) \\ &= \varphi\left(\int_0^\alpha |f'(x)|dx\right) - \varphi\left(\int_0^0 |f'(x)|dx\right) \\ &= \varphi\left(\int_0^\alpha |f'(x)|dx\right) - \varphi(0). \end{aligned} \tag{2.5}$$

We continue the proof by bounding the first term using the convexity of φ. It follows from the Jensen integral inequality applied to the convex function φ and the measure $(1/\alpha)dx$ with $x \in [0, \alpha]$ that

$$\begin{aligned} &\varphi\left(\int_0^\alpha |f'(x)|dx\right) = \varphi\left(\int_0^\alpha \alpha|f'(x)|\left(\frac{1}{\alpha}dx\right)\right) \\ &\leq \int_0^\alpha \varphi\left(\alpha|f'(x)|\right)\left(\frac{1}{\alpha}dx\right) = \frac{1}{\alpha}\int_0^\alpha \varphi\left(\alpha|f'(x)|\right)dx. \end{aligned} \tag{2.6}$$

Joining Equations (2.4), (2.5) and (2.6), we get

$$\int_0^\alpha \varphi'(|f(x) - f(0)|)|f'(x)|dx \leq \frac{1}{\alpha}\int_0^\alpha \varphi\left(\alpha|f'(x)|\right)dx - \varphi(0).$$

This ends the proof. $\square$

This theorem is of particular interest due to its simplicity and the flexibility in the choice of f and φ, with no specific initial conditions required. In a sense, it combines the qualities in [6] and [2, Theorem 2.3] to produce a simpler, more general and more accessible inequality.

Some examples of application of this theorem are below.

Example 1. *If we set $\varphi(x) = x^2$, which is twice differentiable and convex, then Theorem 2.1 gives*

$$\int_0^\alpha 2|f(x) - f(0)||f'(x)|dx \leq \frac{1}{\alpha}\int_0^\alpha (\alpha|f'(x)|)^2\, dx - 0,$$

so

$$\int_0^\alpha |f(x) - f(0)||f'(x)|dx \leq \frac{\alpha}{2}\int_0^\alpha [f'(x)]^2 dx.$$

If we further set $f(0) = 0$, we get the Olech-Opial integral inequality as presented in Theorem 1.1. Our theorem is therefore a valuable generalization.

Example 2. *More generally, if we set $\varphi(x) = x^\beta$ with $\beta > 1$, which is twice differentiable and convex, then Theorem 2.1 gives*

$$\int_0^\alpha \beta|f(x) - f(0)|^{\beta-1}|f'(x)|dx \leq \frac{1}{\alpha}\int_0^\alpha (\alpha|f'(x)|)^\beta\, dx - 0,$$

so

$$\int_0^\alpha |f(x) - f(0)|^{\beta-1}|f'(x)|dx \leq \frac{\alpha^{\beta-1}}{\beta}\int_0^\alpha |f'(x)|^\beta dx.$$

This inequality generalizes that in the previous example, corresponding to the case where $\beta = 2$.

Example 3. *If we set $\varphi(x) = e^{\lambda x}$ with $\lambda \in \mathbb{R}$, which is twice differentiable and convex, then Theorem 2.1 gives*

$$\int_0^\alpha \lambda e^{\lambda|f(x)-f(0)|}|f'(x)|dx \leq \frac{1}{\alpha}\int_0^\alpha e^{\lambda\alpha|f'(x)|}dx - 1.$$

Example 4. *If we set $\varphi(x) = \log(1 + e^{\lambda x})$ with $\lambda \in \mathbb{R}$, which is twice differentiable and convex, then Theorem 2.1 gives*

$$\int_0^\alpha \frac{\lambda e^{\lambda|f(x)-f(0)|}}{1 + e^{\lambda|f(x)-f(0)|}}|f'(x)|dx \leq \frac{1}{\alpha}\int_0^\alpha \log\left(1 + e^{\lambda\alpha|f'(x)|}\right) dx - \log(2).$$

Example 5. *If we set* $\varphi(x) = \cosh(\lambda x)$ *with* $\lambda \in \mathbb{R}$*, which is twice differentiable and convex, then Theorem 2.1 gives*

$$\int_0^\alpha \lambda \sinh\left[\lambda |f(x) - f(0)|\right] |f'(x)| dx \leq \frac{1}{\alpha} \int_0^\alpha \cosh\left[\lambda\alpha |f'(x)|\right] dx - 1.$$

Example 6. *If we set* $\varphi(x) = x^2 + \sin^2(x)$*, which is twice differentiable and convex, then Theorem 2.1 gives*

$$\begin{aligned}&\int_0^\alpha 2\left\{|f(x) - f(0)| + \sin\left[|f(x) - f(0)|\right] \cos\left[|f(x) - f(0)|\right]\right\} |f'(x)| dx \\ \leq &\frac{1}{\alpha} \int_0^\alpha \left\{\alpha^2 [f'(x)]^2 + \sin^2\left[\alpha |f'(x)|\right]\right\} dx.\end{aligned}$$

Many more examples leading to new integral inequalities in the literature can be presented in a similar way.

3. CONCLUSION

This note demonstrates how the scope of the classical Olech-Opial integral inequality can be extended using a convex approach. It is based on the proof methodology developed in [6] and [2, Theorem 2.3]. By introducing an adjustable convex function and removing the initial value restriction, a flexible theorem is provided that can be adapted for use in many areas of analysis. We anticipate that this generalization will inspire further advances and new applications in the theory of integral and differential inequalities.

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