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ON A NEW CONVEX INTEGRAL INEQUALITY INVOLVING THREE FUNCTIONS: THEORY AND EXAMPLES

Christophe Chesneau

ABSTRACT. Convex integral inequalities reveal important links between the values of convex functions and their integrals. In this article, we contribute to the topic by presenting a new convex integral inequality involving three interacting functions and ordered transformations. Several examples demonstrate its applicability and versatility.

1. INTRODUCTION

The concept of convex functions is central to many areas of mathematical analysis. Their formal description is provided below. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$ be such that $a < b$. A function $f : [a, b] \to \mathbb{R}$ is called convex on $[a, b]$ if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, the following inequality is satisfied:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1.1}$$

A key property is that, if f is twice differentiable and convex on $[a, b]$, then f' is non-decreasing. The definition of a convex function has various consequences, including diverse convex integral inequalities. One example is the well-known Hermite-Hadamard integral inequality, which establishes a relationship between

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the average value of a convex function and its boundary values. This inequality forms the basis for numerous extensions, refinements and applications in modern analysis. More comprehensive discussions and generalizations can be found in [1–15].

In this article, we present a new convex integral inequality that involves the interplay of three functions, f, g, and h. The main result is governed by an ordering relation between the auxiliary functions g and h, which induces a corresponding order between the related integral expressions. In particular, if f is convex on $[a, b]$ and the functions g and h satisfy $g(x) \geq h(x)$ for all $x \in [a, b]$ (together with additional conditions specified later), then we have an inequality of the form

$$\int_a^b \mathcal{T}(g)(x)f(x)dx \leq \int_a^b \mathcal{T}(h)(x)f(x)dx,$$

where $\mathcal{T}$ denotes a suitably defined transformation. This result is of particular interest as it establishes a unified framework linking convexity with ordered functional transformations. It thus provides new analytical tools for the study of convex integral inequalities.

The rest of the article is organized as follows: A detailed formulation of the main result is provided in Section 2. Section 3 is devoted to several illustrative examples that demonstrate the applicability and scope of the obtained inequality. Concluding remarks and possible directions for further research are discussed in Section 4.

2. Theorem

We now present the main theorem, which offers a complete and formal statement of the result preliminarily discussed in the introduction.

Theorem 2.1. *Let* $a, b \in \mathbb{R}$ *such that* $b > a \geq 0$. *Let* $f, g, h : [a, b] \to \mathbb{R}$ *be three differentiable functions such that* f *is twice differentiable and convex on* $[a, b]$ *with* $f(0) = 0$,

$$b(g(b) - h(b))f(b) \leq a(g(a) - h(a))f(a),$$

and, for any $x \in [a, b]$,

$$g(x) \geq h(x).$$

Then the following inequality holds:

$$\int_a^b [2g(x) + xg'(x)]\, f(x)dx \leq \int_a^b [2h(x) + xh'(x)]\, f(x)dx.$$

Proof. The proof involves considering the following difference of integrals:

$$\int_a^b [2g(x) + xg'(x)]\, f(x)dx - \int_a^b [2h(x) + xh'(x)]\, f(x)dx$$
$$= \int_a^b [2g(x) + xg'(x) - 2h(x) - xh'(x)]\, f(x)dx$$

and prove that it is non-positive.

Using standard derivatives, developments and a well-calibrated integration by parts, we get

$$\begin{aligned}
&\int_a^b [2g(x) + xg'(x) - 2h(x) - xh'(x)]\, f(x)dx \\
&= \int_a^b \left[2x(g(x) - h(x)) + x^2(g'(x) - h'(x))\right] \frac{f(x)}{x} dx \\
&= \int_a^b \left[x^2(g(x) - h(x))\right]' \frac{f(x)}{x} dx \\
&= \left[x^2(g(x) - h(x))\frac{f(x)}{x}\right]_a^b - \int_a^b x^2(g(x) - h(x))\frac{xf'(x) - f(x)}{x^2} dx \\
&= b(g(b) - h(b))f(b) - a(g(a) - h(a))f(a) \\
&- \int_a^b (g(x) - h(x))(xf'(x) - f(x))dx. \qquad (2.1)
\end{aligned}$$

Let us prove that this last quantity is non-positive.

Since

$$b(g(b) - h(b))f(b) \leq a(g(a) - h(a))f(a),$$

the first main term is non-positive. It remains to prove that the integral expression of the second main term (without the minus sign) is non-negative. Since f is twice differentiable and convex on $[a, b]$, f' is non-decreasing. This and $f(0) = 0$ imply that, for any $x \in [a, b]$,

$$f(x) = f(0) + \int_0^x f'(t)dt = \int_0^x f'(t)dt \leq f'(x) \int_0^x dt = xf'(x),$$

so that

$$xf'(x) - f(x) \geq 0.$$

This combined with, for any $x \in [a, b]$, $g(x) \geq h(x)$, yields

$$(g(x) - h(x))(xf'(x) - f(x))dx \geq 0,$$

so that

$$\int_a^b (g(x) - h(x))(xf'(x) - f(x))dx \geq 0.$$

We thus have

$$\begin{aligned} & b(g(b) - h(b))f(b) - a(g(a) - h(a))f(a) \\ & - \int_a^b (g(x) - h(x))(xf'(x) - f(x))dx \leq 0. \end{aligned} \tag{2.2}$$

Combining Equations (2.1) and (2.2), we obtain

$$\int_a^b \left[2g(x) + xg'(x) - 2h(x) - xh'(x)\right] f(x)dx \leq 0,$$

which implies that

$$\int_a^b \left[2g(x) + xg'(x)\right] f(x)dx \leq \int_a^b \left[2h(x) + xh'(x)\right] f(x)dx.$$

This completes the proof. □

Some remarks on the statement of Theorem 2.1 are provided below.

In the case $b = +\infty$, we have implicitly set

$$b(g(b) - h(b))f(b) = \lim_{x \to +\infty} x(g(x) - h(x))f(x).$$

Note that the assumption

$$b(g(b) - h(b))f(b) \leq a(g(a) - h(a))f(a),$$

is obviously satisfied if $g(a) = h(a)$ and $g(b) = h(b)$, or if $a = 0$ and $g(b) = h(b)$, which both have the advantage of being independent of f.

The transformation $\mathcal{T}(g)$ mentioned in the introduction is thus defined by, for any $x \in [a, b]$,

$$\mathcal{T}(g)(x) = 2g(x) + xg'(x).$$

This definition stems naturally from the theory, but does not appear to have a specific practical application.

To the best of our knowledge, Theorem 2.1 is a new addition to the literature. It builds on convex integral inequalities by incorporating multiple interacting functions and ordered transformations. Furthermore, it offers a versatile framework for deriving additional inequalities. To demonstrate these possibilities, we present several comprehensive examples in the next section.

3. EXAMPLES

For convenience, we focus on the case $a = 0$ and $b = 1$.

Example 1. *Let us set* $a = 0$, $b = 1$, $g(x) = x^{\alpha}$ *and* $h(x) = x^{\beta}$ *with* $\beta \geq \alpha > 0$. *Then we have* $g(0) = h(0) = 0$ *and* $g(1) = h(1) = 1$, *and, for any* $x \in [0, 1]$, $g(x) = x^{\alpha} \geq x^{\beta} = h(x)$. *For a function* $f : [0, 1] \to \mathbb{R}$ *twice differentiable and convex on* $[0, 1]$ *with* $f(0) = 0$, *Theorem 2.1 gives*

$$\int_0^1 [2g(x) + xg'(x)] f(x)dx \leq \int_0^1 [2h(x) + xh'(x)] f(x)dx,$$

so that

$$(2 + \alpha) \int_0^1 x^{\alpha} f(x)dx \leq (2 + \beta) \int_0^1 x^{\beta} f(x)dx.$$

We emphasize the mathematical elegance of this simple integral inequality, which, as far as we are aware, has not been published in any existing literature.

As a numerical example, let us set $\beta = \pi$, $\alpha = e$ *and* $f(x) = e^x - 1$. *Then we have*

$$(2 + e) \int_0^1 x^e(e^x - 1)dx \approx 1.56037 \leq 1.57039 \approx (2 + \pi) \int_0^1 x^{\pi}(e^x - 1)dx.$$

The desired result is obtained.

For another example, let us set $\beta = 3$, $\alpha = 2$ *and* $f(x) = \sqrt{1 + x^2} - 1$. *Then we have*

$$4 \int_0^1 x^2 \left[\sqrt{1 + x^2} - 1\right] dx \approx 0.34730 \leq 0.35948 \approx 5 \int_0^1 x^3 \left[\sqrt{1 + x^2} - 1\right] dx.$$

The expected inequality is satisfied.

Example 2. *Let us set* $a = 0$*,* $b = 1$*,* $g(x) = \log(1 + x)$ *and* $h(x) = x\log(2)$*. Then we have* $g(0) = h(0) = 0$ *and* $g(1) = h(1) = \log(2)$*, and, for any* $x \in [0, 1]$*,* $g(x) = \log(1 + x) \geq x\log(2) = h(x)$*. For a function* $f : [0, 1] \to \mathbb{R}$ *twice differentiable and convex on* $[0, 1]$ *with* $f(0) = 0$*, Theorem 2.1 gives*

$$\int_0^1 [2g(x) + xg'(x)] f(x)dx \leq \int_0^1 [2h(x) + xh'(x)] f(x)dx,$$

so that

$$\int_0^1 \left[2\log(1 + x) + \frac{x}{1 + x}\right] f(x)dx \leq 3\log(2) \int_0^1 xf(x)dx,$$

As a numerical example, let us set $f(x) = e^x - 1$*. Then we have*

$$\int_0^1 \left[2\log(1 + x) + \frac{x}{1 + x}\right] (e^x - 1)dx \approx 1.03102 \leq 1.0397$$

$$\approx 3\log(2) \int_0^1 x(e^x - 1)dx.$$

The expected inequality is obtained.

For another example, let us set $f(x) = \sqrt{1 + x^2} - 1$*. Then we have*

$$\int_0^1 \left[2\log(1 + x) + \frac{x}{1 + x}\right] \left[\sqrt{1 + x^2} - 1\right] dx \approx 0.223445$$

$$\leq 0.22765 \approx 3\log(2) \int_0^1 x \left[\sqrt{1 + x^2} - 1\right] dx.$$

The desired inequality is get.

Example 3. *Let us set* $a = 0$*,* $b = 1$*,* $g(x) = \sin[(\pi/2)x]$ *and* $h(x) = x$*. Then we have* $g(0) = h(0) = 0$ *and* $g(1) = h(1) = 1$*, and, by the Jordan inequality, for any* $x \in [0, 1]$*,* $g(x) = \sin[(\pi/2)x] \geq x = h(x)$*. For a function* $f : [0, 1] \to \mathbb{R}$ *twice differentiable and convex on* $[0, 1]$ *with* $f(0) = 0$*, Theorem 2.1 gives*

$$\int_0^1 [2g(x) + xg'(x)] f(x)dx \leq \int_0^1 [2h(x) + xh'(x)] f(x)dx,$$

so that

$$\int_0^1 \left[2\sin\left(\frac{\pi}{2}x\right) + x\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)\right] f(x)dx \leq 3\int_0^1 xf(x)dx.$$

As a numerical example, let us set $f(x) = e^x - 1$. *Then we have*

$$\int_0^1 \left[2\sin\left(\frac{\pi}{2}x\right) + x\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)\right](e^x - 1)dx \approx 1.4642$$
$$\leq 1.5 = 3\int_0^1 x(e^x - 1)dx.$$

The desired result is obtained.

For another example, let us set $f(x) = \sqrt{1+x^2} - 1$. *Then we have*

$$\int_0^1 \left[2\sin\left(\frac{\pi}{2}x\right) + x\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)\right]\left[\sqrt{1+x^2} - 1\right]dx \approx 0.311772$$
$$\leq 0.32843 \approx 3\int_0^1 x\left[\sqrt{1+x^2} - 1\right]dx.$$

The expected inequality is satisfied.

Example 4. *Let us set* $a = 0$, $b = 1$, $g(x) = x$ *and* $h(x) = 2^x - 1$. *Then we have* $g(0) = h(0) = 0$ *and* $g(1) = h(1) = 1$, *and, by the Bernoulli inequality, for any* $x \in [0,1]$, $g(x) = x \geq 2^x - 1 = h(x)$. *For a function* $f : [0,1] \to \mathbb{R}$ *twice differentiable and convex on* $[0,1]$ *with* $f(0) = 0$, *Theorem 2.1 gives*

$$\int_0^1 [2g(x) + xg'(x)]f(x)dx \leq \int_0^1 [2h(x) + xh'(x)]f(x)dx,$$

so that

$$3\int_0^1 xf(x)dx \leq \int_0^1 [2(2^x - 1) + \log(2)x2^x]f(x)dx.$$

As a numerical example, let us set $f(x) = e^x - 1$. *Then we have*

$$3\int_0^1 x(e^x - 1)dx = 1.5 \leq 1.51428 \approx \int_0^1 [2(2^x - 1) + \log(2)x2^x](e^x - 1)dx.$$

The expected inequality is get.

For another example, let us set $f(x) = \sqrt{1+x^2} - 1$. *Then we have*

$$3\int_0^1 x\left[\sqrt{1+x^2} - 1\right]dx \approx 0.32843$$
$$\leq 0.335134 \approx \int_0^1 [2(2^x - 1) + \log(2)x2^x]\left[\sqrt{1+x^2} - 1\right]dx.$$

The desired result is obtained.

Example 5. *Let us set* $a = 0$*,* $b = 1$*,* $g(x) = x$ *and* $h(x) = (e^x - 1)/(e - 1)$*. Then we have* $g(0) = h(0) = 0$ *and* $g(1) = h(1) = 1$*, and, for any* $x \in [0, 1]$*,* $g(x) = x \geq (e^x - 1)/(e - 1) = h(x)$*. For a function* $f : [0, 1] \to \mathbb{R}$ *twice differentiable and convex on* $[0, 1]$ *with* $f(0) = 0$*, Theorem 2.1 gives*

$$\int_0^1 [2g(x) + xg'(x)]\, f(x)dx \leq \int_0^1 [2h(x) + xh'(x)]\, f(x)dx,$$

so that

$$3\int_0^1 xf(x)dx \leq \int_0^1 \left(2\frac{e^x - 1}{e - 1} + \frac{xe^x}{e - 1}\right) f(x)dx.$$

As a numerical example, let us set $f(x) = e^x - 1$*. Then we have*

$$3\int_0^1 x(e^x - 1)dx = 1.5 \leq 1.5208 \approx \int_0^1 \left(2\frac{e^x - 1}{e - 1} + \frac{xe^x}{e - 1}\right)(e^x - 1)dx.$$

The desired inequality is satisfied.

For another example, let us set $f(x) = \sqrt{1 + x^2} - 1$*. Then we have*

$$\begin{aligned} &3\int_0^1 x\left[\sqrt{1 + x^2} - 1\right] dx \approx 0.32843 \\ &\leq 0.338159 \approx \int_0^1 \left(2\frac{e^x - 1}{e - 1} + \frac{xe^x}{e - 1}\right)\left[\sqrt{1 + x^2} - 1\right] dx. \end{aligned}$$

The expected inequality is get.

Many more examples can be given to illustrate the sharpness and versatility of Theorem 2.1.

4. Conclusion

In this article, we introduced a new convex integral inequality involving three interacting functions and ordered transformations. The inequality was shown to be versatile by a number of illustrative examples. Our findings pave the way for further research, including multidimensional generalizations, operator-based extensions and applications in optimization theory and other areas of applied analysis.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CAEN-NORMANDIE
UFR DES SCIENCES - CAMPUS 2, CAEN
FRANCE.
Email address: christophe.chesneau@gmail.com