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SOME NEW YOUNG-TYPE INTEGRAL INEQUALITIES DERIVED FROM COMPOSITION FUNCTION APPROACHES

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ABSTRACT. We present new results combining composition function approaches with Young-type integral inequalities. These approaches consider compositions of functions within the integrands and compositions of the main integrals. The proposed framework involves two functions that serve different purposes. Complete proofs are provided for all the main results.

1. INTRODUCTION

Inequalities occupy a central position in mathematical analysis. In particular, they are an indispensable means of obtaining bounds, comparing quantities and revealing structural relationships between functions. Comprehensive expositions, illustrative examples and applications of numerous inequalities can be found in [2, 3, 15, 17, 30]. One of the most renowned classical results in this area is the Young integral inequality, first introduced by W.H. Young in [33]. This inequality elegantly describes the connection between a function, f, and its inverse, f^{-1}, through their integral formulations. The precise statement is presented in the theorem below.

Theorem 1.1. *Let $f : [0, +\infty) \to [0, +\infty)$ be a continuous (strictly) increasing function satisfying $f(0) = 0$. Let f^{-1} be the inverse function of f. Then, for any*

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$a, b \geq 0$ *provided that the relevant integrals are well defined, we have*

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab.$$

In recent years, there has been significant progress in extending, sharpening, and applying the Young integral inequality in various contexts. For further developments and detailed surveys, the reader is referred to [1, 4–14, 16, 18–29, 31, 32, 34].

This article introduces new Young-type integral inequalities obtained through two complementary approaches: one based on composing functions within the integrands, and the other on composing the main integrals. Each approach involves two distinct functions, f and g, together with their respective inverses, f^{-1} and g^{-1}. Using integral techniques such as changes of variables and the Jensen integral inequality, we derive a new class of Young-type integral inequalities. These results are of interest because they provide a unified framework with potential applications in analysis, probability theory, optimization and functional inequalities.

The remainder of the article is organized as follows: In Section 2, we formulate a first theorem addressing Young-type integral inequalities involving compositions of functions within the integrands, and provide a proof. In Section 3, we proceed similarly but with compositions of the main integrals. Section 4 discusses an additional theorem on the same topic but within a different framework. Finally, Section 5 contains concluding remarks.

2. First theorem

The theorem below presents three Young-type integral inequalities. They are derived from the Young integral inequality and changes of variables.

Theorem 2.1. *Let* $f, g : [0, +\infty) \to [0, +\infty)$ *be two continuous (strictly) increasing functions satisfying* $f(0) = 0$ *and* $g(0) = 0$*. Let* f^{-1} *and* g^{-1} *be the inverse functions of* f *and* g*, respectively. Then, for any* $a, b \geq 0$ *provided that the relevant integrals are well defined,*

(1) *we have*

$$\int_0^a g(f(x))dx + \int_0^b f^{-1}(g^{-1}(x))dx \geq ab,$$

(2) *we have*

$$\int_0^a g(f(x))dx + \int_0^b f^{-1}(x)g'(x)dx \geq ag(b),$$

(3) *we have*

$$\int_0^a g(x)\frac{1}{f'(f^{-1}(x))}dx + \int_0^b f^{-1}(x)g'(x)dx \geq f^{-1}(a)g(b).$$

Proof. (1) Let us introduce the composition function $h = g(f)$. Then h^{-1} satisfies $h(h^{-1}(x)) = h^{-1}(h(x)) = x$ for any $x \in [0, +\infty)$. Some basic manipulations give

$$h^{-1} = f^{-1}(g^{-1}).$$

Applying the Young integral inequality to the function h, we obtain

$$\int_0^a g(f(x))dx + \int_0^b f^{-1}(g^{-1}(x))dx$$
$$= \int_0^a h(x)dx + \int_0^b h^{-1}(x)dx \geq ab.$$

(2) Making the change of variables $x = g^{-1}(y)$ in the second integral using $g(0) = 0$, and applying the Young integral inequality to the function $h = g(f)$ and the upper limits a and $g(b)$, we get

$$\int_0^a g(f(x))dx + \int_0^b f^{-1}(x)g'(x)dx$$
$$= \int_0^a g(f(x))dx + \int_{g(0)}^{g(b)} f^{-1}(g^{-1}(y))dy$$
$$= \int_0^a h(x)dx + \int_0^{g(b)} h^{-1}(x)dx \geq ag(b).$$

(3) Making the change of variables $x = f(y)$ in the first integral using $f^{-1}(0) = 0$ and $x = g^{-1}(z)$ in the second integral using $g(0) = 0$, and applying the Young integral inequality to the function $h = g(f)$ and the

upper limits $f^{-1}(a)$ and $g(b)$, we obtain

$$\int_0^a g(x)\frac{1}{f'(f^{-1}(x))}dx + \int_0^b f^{-1}(x)g'(x)dx$$
$$= \int_{f^{-1}(0)}^{f^{-1}(a)} g(f(y))dy + \int_{g(0)}^{g(b)} f^{-1}(g^{-1}(z))dz$$
$$= \int_0^{f^{-1}(a)} h(x)dx + \int_0^{g(b)} h^{-1}(x)dx \geq f^{-1}(a)g(b).$$

This completes the proof of the theorem. □

By setting $g(x) = x$, item 1 of Theorem 2.1 reduces to the Young integral inequality. More generally, this theorem demonstrates how the Young integral inequality can be naturally extended by incorporating a composition function scheme within the integrands. This extension is made possible through carefully chosen changes of variables and integral manipulations, highlighting the flexibility and generality of the composition approach.

3. Second theorem

The theorem below presents three Young-type integral inequalities. They are derived from the Young integral inequality and the Jensen integral inequality.

Theorem 3.1. *Let $f, g : [0, +\infty) \to [0, +\infty)$ be two continuous (strictly) increasing functions satisfying $f(0) = 0$ and $g(0) = 0$. Let f^{-1} and g^{-1} be the inverse functions of f and g, respectively. Then, for any $a, b \geq 0$ provided that the relevant integrals are well defined,*

(1) *assuming that g is concave, we have*

$$ag\left(\frac{1}{a}\int_0^a f(x)dx\right) + \int_0^b f^{-1}(g^{-1}(x))dx \geq ab,$$

(2) *assuming that f^{-1} is concave, we have*

$$\int_0^a g(f(x))dx + bf^{-1}\left(\frac{1}{b}\int_0^b g^{-1}(x)dx\right) \geq ab,$$

(3) *assuming that g and f^{-1} are concave, we have*

$$ag\left(\frac{1}{a}\int_0^a f(x)dx\right) + bf^{-1}\left(\frac{1}{b}\int_0^b g^{-1}(x)dx\right) \geq ab.$$

Proof.

(1) Assuming that g is concave, applying the Jensen integral inequality to g on the interval $[0, a]$, we get

$$g\left(\frac{1}{a}\int_0^a f(x)dx\right) \geq \frac{1}{a}\int_0^a g(f(x))dx.$$

Using this and applying the Young integral inequality to the function $h = g(f)$, we obtain

$$\begin{aligned} & ag\left(\frac{1}{a}\int_0^a f(x)dx\right) + \int_0^b f^{-1}(g^{-1}(x))dx \\ \geq\ & a\frac{1}{a}\int_0^a g(f(x))dx + \int_0^b f^{-1}(g^{-1}(x))dx \\ =\ & \int_0^a h(x)dx + \int_0^b h^{-1}(x)dx \geq ab. \end{aligned}$$

(2) Assuming that f^{-1} is concave, applying the Jensen integral inequality to f^{-1} on the interval $[0, b]$, we get

$$f^{-1}\left(\frac{1}{b}\int_0^b g^{-1}(x)dx\right) \geq b\frac{1}{b}\int_0^b f^{-1}(g^{-1}(x))dx.$$

Using this and applying the Young integral inequality to the function $h = g(f)$, we obtain

$$\begin{aligned} & \int_0^a g(f(x))dx + bf^{-1}\left(\frac{1}{b}\int_0^b g^{-1}(x)dx\right) \\ \geq\ & \int_0^a g(f(x))dx + b\frac{1}{b}\int_0^b f^{-1}(g^{-1}(x))dx \\ =\ & \int_0^a h(x)dx + \int_0^b h^{-1}(x)dx \geq ab. \end{aligned}$$

(3) Assuming that g is concave, the Jensen integral inequality applied to the interval $[0, a]$ ensures that

$$g\left(\frac{1}{a}\int_0^a f(x)dx\right) \geq \frac{1}{a}\int_0^a g(f(x))dx$$

Similarly, since f^{-1} is concave, we have

$$f^{-1}\left(\frac{1}{b}\int_0^b g^{-1}(x)dx\right) \geq \frac{1}{b}\int_0^b f^{-1}(g^{-1}(x))dx.$$

Combining these inequalities and applying the Young integral inequality to the function $h = g(f)$, we get

$$\begin{aligned}
&ag\left(\frac{1}{a}\int_0^a f(x)dx\right) + bf^{-1}\left(\frac{1}{b}\int_0^b g^{-1}(x)dx\right)\\
&= a\frac{1}{a}\int_0^a g(f(x))dx + b\frac{1}{b}\int_0^b f^{-1}(g^{-1}(x))dx\\
&= \int_0^a h(x)dx + \int_0^b h^{-1}(x)dx \geq ab.
\end{aligned}$$

This concludes the proof of the theorem. □

By setting $g(x) = x$, item 1 of Theorem 3.1 reduces to the Young integral inequality. More generally, this theorem illustrates how the Young integral inequality can be naturally extended by incorporating a composition function scheme of the main integrals. This extension is achieved through the application of the Jensen integral inequality combined with careful integral manipulations, emphasizing both the flexibility and generality of the composition approach.

4. Supplementary theorem

The theorem below presents a Young-type integral inequality derived from the Young integral inequality through a specific composition function scheme.

Theorem 4.1. *Let $f, g : [0, +\infty) \to [0, +\infty)$ be two continuous (strictly) increasing functions satisfying $f(0) = 0$ and $g(0) = 0$. Let f^{-1} and g^{-1} be the inverse functions of f and g, respectively. Then, for any $a, b \geq 0$ provided that the relevant integrals are well defined, we have*

$$\int_0^{e^{\int_0^a g(x)dx}} f(x)dx + \int_0^{e^{\int_0^b g^{-1}(x)dx}} f^{-1}(x)dx \geq e^{ab}.$$

Proof. Let us set

$$a_* = e^{\int_0^a g(x)dx}, \quad b_* = e^{\int_0^b g^{-1}(x)dx}.$$

Applying the Young integral inequality to the function f, and the upper limits a_* and b_*, using basic properties of the exponential function, and applying the Young integral inequality to the function g, and the upper limits a and b (into the exponential), we obtain

$$\int_0^{e^{\int_0^a g(x)dx}} f(x)dx + \int_0^{e^{\int_0^b g^{-1}(x)dx}} f^{-1}(x)dx = \int_0^{a_*} f(x)dx + \int_0^{b_*} f^{-1}(x)dx$$
$$\geq a_* b_* = e^{\int_0^a g(x)dx} e^{\int_0^b g^{-1}(x)dx} = e^{\int_0^a g(x)dx + \int_0^b g^{-1}(x)dx} \geq e^{ab}.$$

This ends the proof of the theorem. □

In the setting of Theorem 4.1, changing the roles of f and f^{-1}, the following inequality can be proved in a similar way:

$$\int_0^{e^{\int_0^b g^{-1}(x)dx}} f(x)dx + \int_0^{e^{\int_0^a g(x)dx}} f^{-1}(x)dx \geq e^{ab}.$$

An alternative formulation of Theorem 4.1, which preserves the product ab as a lower bound, is given by the following inequality:

$$\log\left(\int_0^{e^{\int_0^a g(x)dx}} f(x)dx + \int_0^{e^{\int_0^b g^{-1}(x)dx}} f^{-1}(x)dx\right) \geq ab.$$

Based on this formulation, it is natural to consider a more general inequality of the form

$$\phi^{-1}\left(\int_0^{\phi\left(\int_0^a g(x)dx\right)} f(x)dx + \int_0^{\phi\left(\int_0^b g^{-1}(x)dx\right)} f^{-1}(x)dx\right) \geq ab, \tag{4.1}$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is a function whose precise properties are to be determined in future work.

5. Conclusion

In this article, we presented new Young-type integral inequalities developed using composition function schemes. These extend classical results and provide a unified framework for analysis. The main theorems presented highlight the interplay between two functions and their inverses within integral inequalities, alongside detailed proofs. These results pave the way for further exploration, including potential generalizations to multidimensional settings, as well as applications in optimization, probability and functional inequalities. Future research may also investigate sharper bounds, alternative composition schemes (such as that in Equation (4.1)) and connections with other integral inequalities in mathematical analysis.

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