J COMP SCI
APPL MATH
Journal of Computer Science and Applied Mathematics
Vol. 8, no.1, (2026), 31–37
ISSN: 1857-9582 https://doi.org/10.37418/jcsam.8.1.4

EXTENDING A SPECIFIC INTEGRAL INEQUALITY BY INCORPORATING A WEIGHT FUNCTION

Christophe Chesneau

ABSTRACT. This note introduces a new integral inequality, extending the scope of an existing result by incorporating a weight function and relaxing the continuity assumption. A complete proof is provided, alongside an examination of several special cases.

1. INTRODUCTION

Integral inequalities form the basis of mathematical analysis. They provide essential tools for establishing bounds in a variety of functional and differential problems. These inequalities often arise in fields such as approximation theory, convex analysis and the study of special functions, where precise control over the magnitude of integrals is vital. Comprehensive treatments of classical results on these inequalities can be found in the books [1, 2, 10, 13, 23, 24]. In recent years, significant progress has been made in extending these inequalities, including the development of new kernel functions, weighted forms, and generalizations to broader functional spaces. Notable contributions and recent advances are documented in [3–7, 11, 14–16, 18–22].

The discovery of new integral inequalities strengthens the theoretical framework of the subject, expanding its applicability to modern analysis, numerical

Key words and phrases. Integral inequalities, weight functions, mean value theorem for integrals.
Submitted: 15.01.2026; *Accepted:* 31.01.2026; *Published:* 11.02.2026.

methods and the study of differential and integral equations. A few years ago, the integral inequality presented in the theorem below was discussed on the mathematical forum *Math.StackExchange.com*, attracting some interest within the mathematical community.

Theorem 1.1. *[12] Let $f, g : [0,1] \to [0,1]$ be continuous functions. We suppose that f is non-decreasing. Then we have*

$$\int_0^1 f(g(x))dx \leq \int_0^1 g(x)dx + \int_0^1 f(x)dx.$$

A proof was provided by [17] using the mean value theorem for integrals. In this note, we extend this result by introducing a weight function, w, relaxing the continuity assumption and providing an alternative proof. We also discuss several special cases of the new weighted integral inequality.

The main result, together with its proof and special cases, is presented in Section 2. Section 3 provides concluding remarks.

2. Result and special cases

2.1. **Result.** The theorem below can be considered a weighted extension of Theorem 1.1. It should also be noted that the functions g and w are not required to be continuous.

Theorem 2.1. *Let $f, g : [0,1] \to [0,1]$ and $w : [0,1] \to [0,+\infty)$ be functions. We suppose that f is non-decreasing. Then we have*

$$\int_0^1 f(g(x))w(x)dx \leq \int_0^1 g(x)w(x)dx + \left(\int_0^1 f(x)dx\right)\left(\int_0^1 w(x)dx\right),$$

provided that the integrals exist.

Proof. We begin by examining the difference

$$\int_0^1 f(g(x))w(x)dx - \int_0^1 g(x)w(x)dx = \int_0^1 \left(f(g(x)) - g(x)\right) w(x)dx. \tag{2.1}$$

Since $f(g(x)) \in [0,1]$, and g and w are non-negative, we have

$$\int_0^1 (f(g(x)) - g(x))\, w(x)dx \leq \int_0^1 (f(g(x)) - g(x)f(g(x)))\, w(x)dx$$

$$= \int_0^1 (1 - g(x))\, f(g(x))w(x)dx = \int_0^1 \left(\int_{g(x)}^1 du\right) f(g(x))w(x)dx. \tag{2.2}$$

Since f is non-decreasing, for any $u \in [g(x), 1]$, we have $f(g(x)) \leq f(u)$. This, combined with the facts that f, g and w are non-negative, gives

$$\int_0^1 \left(\int_{g(x)}^1 du\right) f(g(x))w(x)dx \leq \int_0^1 \left(\int_{g(x)}^1 f(u)du\right) w(x)dx$$

$$\leq \int_0^1 \left(\int_0^1 f(u)du\right) w(x)dx = \left(\int_0^1 f(x)dx\right) \left(\int_0^1 w(x)dx\right). \tag{2.3}$$

Combining Equations (2.1), (2.2) and (2.3), we get

$$\int_0^1 f(g(x))w(x)dx - \int_0^1 g(x)w(x)dx \leq \left(\int_0^1 f(x)dx\right) \left(\int_0^1 w(x)dx\right),$$

which implies that

$$\int_0^1 f(g(x))w(x)dx \leq \int_0^1 g(x)w(x)dx + \left(\int_0^1 f(x)dx\right) \left(\int_0^1 w(x)dx\right).$$

This completes the proof. □

Setting $w(x) = 1$, Theorem 2.1 reduces to Theorem 1.1. We emphasize that the proof does not rely on the continuity of the functions involved or the mean value theorem for integrals. Furthermore, dealing with the weight function w is quite natural.

The theorem below provides a refinement of Theorem 2.1 for the case where g is non-decreasing and admits an inverse function g^{-1}.

Theorem 2.2. *Let $f, g : [0, 1] \to [0, 1]$ and $w : [0, 1] \to [0, +\infty)$ be functions. We suppose that f and g are non-decreasing and that g^{-1} exists. Then we have*

$$\int_0^1 f(g(x))w(x)dx \leq \int_0^1 g(x)w(x)dx + \int_0^1 f(x)h(x)dx,$$

where

$$h(x) = \int_0^{g^{-1}(x)} w(t)dt,$$

provided that the integrals exist.

Proof. We can reuse Equations (2.1) and (2.2), but Equation (2.3) requires the modification described below. Since the integrand is non-negative, the order of integration can be interchanged by applying the Fubini-Tonelli theorem, yielding

$$\int_0^1 \left(\int_{g(x)}^1 du \right) f(g(x))w(x)dx \leq \int_0^1 \left(\int_{g(x)}^1 f(u)du \right) w(x)dx$$

$$= \int_0^1 \left(\int_0^{g^{-1}(u)} w(x)dx \right) f(u)du = \int_0^1 f(x)h(x)dx. \tag{2.4}$$

Combining Equations (2.1), (2.2) and (2.4), we get

$$\int_0^1 f(g(x))w(x)dx - \int_0^1 g(x)w(x)dx \leq \int_0^1 f(x)h(x)dx,$$

which implies that

$$\int_0^1 f(g(x))w(x)dx \leq \int_0^1 g(x)w(x)dx + \int_0^1 f(x)h(x)dx.$$

This concludes the proof. □

In certain situations, Theorem 2.2 may be preferable to Theorem 2.1, since it provides a sharper result at the cost of a more technical upper bound.

2.2. **Special cases.** Some notable special cases of Theorem 2.1 are presented below. The integrals involved are assumed to exist.

- Setting $g = f$, Theorem 2.1 gives

$$\int_0^1 (f \circ f)(x)w(x)dx \leq \int_0^1 f(x)w(x)dx + \left(\int_0^1 f(x)dx \right) \left(\int_0^1 w(x)dx \right).$$

Furthermore, setting $w = f$ yields

$$\int_0^1 (f \circ f)(x)f(x)dx \leq \int_0^1 f^2(x)dx + \left(\int_0^1 f(x)dx \right)^2.$$

This inequality is particularly elegant due to the symmetry it exhibits between the function f and its composition $f \circ f$, and the way it balances the integral of the squared function with the square of the integral.

- Setting $g = f^{-1}$, Theorem 2.1 gives

$$\int_0^1 xw(x)dx \leq \int_0^1 f^{-1}(x)w(x)dx + \left(\int_0^1 f(x)dx\right)\left(\int_0^1 w(x)dx\right).$$

This is reminiscent of the Young integral inequality, in that it involves a function f and its inverse. However, the results obtained here are substantially different. For further details on the Young integral inequality, see, for instance, [8, 9], and the references therein.

- Setting $w = g$, Theorem 2.1 gives

$$\int_0^1 f(g(x))g(x)dx \leq \int_0^1 g^2(x)dx + \left(\int_0^1 f(x)dx\right)\left(\int_0^1 g(x)dx\right).$$

- Setting $w(x) = x^\alpha$ with $\alpha \geq 0$, Theorem 2.1 yields

$$\int_0^1 f(g(x))x^\alpha dx \leq \int_0^1 g(x)x^\alpha dx + \left(\int_0^1 f(x)dx\right)\left(\int_0^1 x^\alpha dx\right)$$
$$= \int_0^1 g(x)x^\alpha dx + \frac{1}{\alpha+1}\int_0^1 f(x)dx.$$

In particular, for $\alpha = 0$, this result reduces to Theorem 1.1.

- Setting $g(x) = x^\alpha$ with $\alpha \geq 0$, Theorem 2.1 gives

$$\int_0^1 f(x^\alpha)w(x)dx \leq \int_0^1 x^\alpha w(x)dx + \left(\int_0^1 f(x)dx\right)\left(\int_0^1 w(x)dx\right).$$

- Setting $f(x) = x^\alpha$ with $\alpha \geq 0$, Theorem 2.1 yields

$$\int_0^1 g^\alpha(x)w(x)dx \leq \int_0^1 g(x)w(x)dx + \left(\int_0^1 x^\alpha dx\right)\left(\int_0^1 w(x)dx\right)$$
$$= \int_0^1 g(x)w(x)dx + \frac{1}{\alpha+1}\int_0^1 w(x)dx.$$

To the best of the knowledge of the author, each of these inequalities is new to the literature.

To end this section, a notable special case of Theorem 2.2 is presented below. Setting $f(x) = 1$ and $w(x) = 1$, Theorem 2.2 gives

$$1 \leq \int_0^1 g(x)dx + \int_0^1 g^{-1}(x)dx. \tag{2.5}$$

This can be presented as a special case of the Young integral inequality, where the upper limits are equal to 1.

3. Concluding remarks

In this note, we have extended a classical integral inequality by introducing a weight function and relaxing the continuity assumption. This provides an alternative proof that is independent of the mean value theorem for integrals. The weighted inequality and its special cases demonstrate the flexibility and potential of such extensions in controlling integral expressions. Future work could involve exploring links existing with the Young integral inequality exhibited in Equation (2.5), or further generalizations in multidimensional domains, with variable functions. Applications could include fractional calculus and dynamic equations. This would open up new avenues for both theoretical development and practical implementation.

Acknowledgment

The author would like to thank the reviewers for their constructive comments.

References

[1] D. Bainov, P. Simeonov: *Integral Inequalities and Applications*, Mathematics and Its Applications, Vol. 57, Kluwer Academic, Dordrecht, 1992.

[2] E.F. Beckenbach, R. Bellman: *Inequalities*, Springer, Berlin, 1961.

[3] B. Benaissa, H. Budak: *On Hardy-type integral inequalities with negative parameter*, Turkish J. Inequal., **5** (2021), 42–47.

[4] B. Benaissa, A. Senouci: *New integral inequalities relating to a general integral operator through monotone functions*, Sahand Commun. Math. Anal., **19** (2022), 41–56.

[5] C. Chesneau: *A novel multivariate integral ratio operator: theory and applications including inequalities*, Asian J. Math. Appl., **2024** (2024), 1–37.

[6] C. Chesneau: *A generalization of the Du integral inequality*, Trans. J. Math. Anal. Appl., **12** (2024), 45–52.

[7] C. Chesneau: *Integral inequalities under diverse parametric primitive exponential-weighted integral inequality assumptions*, Ann. Com. Math., **8** (2025), 43–56.

[8] C. Chesneau: *New Young-type integral inequalities using composition schemes*, Math. Slovaca, **75**(6) (2025), 1311–1330.

[9] C. CHESNEAU: *A new generalization of the Young integral inequality*, Adv. Math. Sci. J., **15**(1) (2026), 111-120.

[10] Z. CVETKOVSKI: *Inequalities: Theorems, Techniques and Selected Problems*, Springer, Berlin Heidelberg, 2012.

[11] W.-S. DU: *New integral inequalities and generalizations of Huang-Du's integral inequality*, Appl. Math. Sci., **17** (2023), 265–272.

[12] L. EPPES (PSEUDONYM): (https://math.stackexchange.com/users/49660/larry-eppes), Prove the following integral inequality: $\int_0^1 f(g(x))dx \leq \int_0^1 f(x)dx + \int_0^1 g(x)dx$, URL (version: 2016-01-13): https://math.stackexchange.com/q/276722

[13] G.H. HARDY, J.E. LITTLEWOOD, G. PÓLYA: *Inequalities*, Cambridge Univ. Press, Cambridge, 1934.

[14] H. HUANG, W.-S. DU: *On a new integral inequality: generalizations and applications*, Axioms, **11** (2022), 1–9.

[15] T.F. MÓRI: *A general inequality of Ngo-Thang-Dat-Tuan type*, J. Inequal. Pure Appl. Math., **10** (2009), 1–11.

[16] Q.A. NGO, D.D. THANG, T.T. DAT, D.A. TUAN: *Notes on an integral inequality*, J. Pure Appl. Math., **7** (2006), 1–5.

[17] RIEMANN (PSEUDONYM): (https://math.stackexchange.com/users/27899/riemann), Prove the following integral inequality: $\int_0^1 f(g(x))dx \leq \int_0^1 f(x)dx + \int_0^1 g(x)dx$, URL (version: 2024-03-27): https://math.stackexchange.com/q/276992

[18] A. SENOUCI, B. BENAISSA, M. SOFRANI: *Some new integral inequalities for negative summation parameters*, Surv. Math. Appl., **18** (2023), 123–133.

[19] W.T. SULAIMAN: *Notes on integral inequalities*, Demonstr. Math., **41** (2008), 887–894.

[20] W.T. SULAIMAN: *New several integral inequalities*, Tamkang J. Math., **42** (2011), 505–510.

[21] W.T. SULAIMAN: *Several ideas on some integral inequalities*, Adv. Pure Math., **1** (2011), 63–66.

[22] W.T. SULAIMAN: *A study on several new integral inequalities*, South Asian J. Math., **42** (2012), 333–339.

[23] W. WALTER: *Differential and Integral Inequalities*, Springer, Berlin, 1970.

[24] B.C. YANG: *Hilbert-Type Integral Inequalities*, Bentham Science Publishers, United Arab Emirates, 2009.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CAEN-NORMANDIE
UFR DES SCIENCES - CAMPUS 2, CAEN
FRANCE.
Email address: christophe.chesneau@gmail.com