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ON THREE REFERENCED TWO-PARAMETER TRIGONOMETRIC IMPROPER INTEGRALS EQUAL TO ZERO

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ABSTRACT. This paper focuses on three referenced integrals that share the following features: (i) they are improper; (ii) they depend on two parameters; (iii) they are of trigonometric type; (iv) they evaluate to zero, making them particularly notable; and (v) their proofs do not require intermediate integrals to be determined, instead relying on the application of trigonometric formulas and various integration techniques. All details are provided.

1. INTRODUCTION

There are numerous integral formulas, many of which are collected in [6]. However, this reference does not provide proofs. In fact, it appears that the proofs for some of these formulas have been lost over time, making it challenging to interpret certain results. Considerable effort has recently been devoted to reconstructing the proofs of such formulas, and in some cases deriving new ones. See, for example, [1–5, 7–10].

In this paper, we examine three specific integral formulas from [6]. Each integral is improper and depends on two parameters. They also involve trigonometric functions and evaluate to zero. This is notable because the integrands possess

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a pronounced oscillatory structure, which makes these results non-trivial. The aim of this work is to present complete and accessible proofs of these formulas. A distinctive feature of our approach is that the proofs do not rely on evaluating known integrals; instead, they are primarily based on trigonometric formulas and standard integration techniques. Full details of all derivations are provided.

The next section states and proves the three integral formulas. Section 3 offers concluding remarks.

2. Results

The three mentioned formulas are presented below, a subsection being devoted to each of them.

2.1. **First formula.**

The first formula is presented in the proposition below. It corresponds to [6, Entry 3.7412 (third result)].

Proposition 2.1. *Let $b > a \geq 0$. Then we have*

$$\int_0^{+\infty} \sin(ax)\cos(bx)\frac{1}{x}dx = 0.$$

Proof. A standard trigonometric formula gives

$$\sin(ax)\cos(bx) = \frac{1}{2}\left(\sin\left((a+b)x\right) - \sin\left((b-a)x\right)\right).$$

Using this and the convergence of the involved integrals, we have

$$\begin{aligned}
&\int_0^{+\infty} \sin(ax)\cos(bx)\frac{1}{x}dx \\
&= \int_0^{+\infty} \frac{1}{2}\left(\sin\left((a+b)x\right) - \sin\left((b-a)x\right)\right)\frac{1}{x}dx \\
&= \frac{1}{2}\left(\int_0^{+\infty} \sin\left((a+b)x\right)\frac{1}{x}dx - \int_0^{+\infty} \sin\left((b-a)x\right)\frac{1}{x}dx\right). \qquad (2.1)
\end{aligned}$$

Making the changes of variables $y = (a+b)x$ and $z = (b-a)x$, with $z > 0$ because $b > a \geq 0$ (this plays a role in the definition of the limits of integration), we have

$$
\begin{aligned}
&\frac{1}{2}\left(\int_0^{+\infty} \sin\left((a+b)x\right)\frac{1}{x}dx - \int_0^{+\infty} \sin\left((b-a)x\right)\frac{1}{x}dx\right)\\
&= \frac{1}{2}\left(\int_0^{+\infty} \sin(y)\frac{1}{y/(a+b)} \times \frac{1}{a+b}dy\right.\\
&\quad \left. - \int_0^{+\infty} \sin(z)\frac{1}{z/(b-a)} \times \frac{1}{b-a}dz\right)\\
&= \frac{1}{2}\left(\int_0^{+\infty} \sin(x)\frac{1}{x}dx - \int_0^{+\infty} \sin(x)\frac{1}{x}dx\right) = 0.
\end{aligned} \tag{2.2}
$$

It follows from Equations (2.1) and (2.2) that

$$\int_0^{+\infty} \sin(ax)\cos(bx)\frac{1}{x}dx = 0.$$

This concludes the proof. □

This proof is interesting because it does not require intermediate integrals to be calculated. It also highlights the significance of the condition $b > a \geq 0$ in the change of variables step. The result also shows that the oscillatory functions $\sin(ax)/x$ and $\cos(bx)$ are, in a sense, orthogonal over $(0, +\infty)$. This can be interpreted as the cancellation of positive and negative contributions of the integrand over this infinite interval.

2.2. **Second formula.** The second formula is presented in the proposition below. It corresponds to [6, Entry 3.7863 (second result)].

Proposition 2.2. *Let $b \geq a > 0$. Then we have*

$$\int_0^{+\infty} (1-\cos(ax))\cos(bx)\frac{1}{x^2}dx = 0.$$

Proof. Let us set

$$I(a) = \int_0^{+\infty} (1-\cos(ax))\cos(bx)\frac{1}{x^2}dx.$$

We start by looking at the special case $b = a > 0$ and then derive the more general case $b \geq a > 0$.

- Case $b = a > 0$. We want to prove that $I(b) = 0$. We have

$$I(b) = \int_0^{+\infty} (1 - \cos(bx)) \cos(bx) \frac{1}{x^2} dx$$

$$(2.3) \qquad = \int_0^{+\infty} \left(\cos(bx) - \cos^2(bx)\right) \frac{1}{x^2} dx.$$

A standard trigonometric formula gives

$$\cos^2(bx) = \frac{1}{2}(1 + \cos(2bx)).$$

Using this and the convergence of the involved integrals, we get

$$\int_0^{+\infty} \left(\cos(bx) - \cos^2(bx)\right) \frac{1}{x^2} dx$$

$$(2.4) \qquad = \int_0^{+\infty} \left(\cos(bx) - \frac{1}{2}(1 + \cos(2bx))\right) \frac{1}{x^2} dx$$

$$= \int_0^{+\infty} \left(\frac{1}{2}(1 - \cos(2bx)) - (1 - \cos(bx))\right) \frac{1}{x^2} dx$$

$$= \frac{1}{2} \int_0^{+\infty} (1 - \cos(2bx)) \frac{1}{x^2} dx - \int_0^{+\infty} (1 - \cos(bx)) \frac{1}{x^2} dx.$$

Making the changes of variables $y = 2bx$ and $z = bx$, we have

$$\frac{1}{2} \int_0^{+\infty} (1 - \cos(2bx)) \frac{1}{x^2} dx - \int_0^{+\infty} (1 - \cos(bx)) \frac{1}{x^2} dx$$

$$(2.5) \qquad = \frac{1}{2} \int_0^{+\infty} (1 - \cos(y)) \frac{1}{(y/(2b))^2} \times \frac{1}{2b} dy$$

$$- \int_0^{+\infty} (1 - \cos(z)) \frac{1}{(z/b)^2} \times \frac{1}{b} dz$$

$$= b \left(\int_0^{+\infty} (1 - \cos(x)) \frac{1}{x} dx - \int_0^{+\infty} (1 - \cos(x)) \frac{1}{x} dx \right) = 0.$$

It follows from Equations (2.3), (2.4) and (2.5) that

$$I(b) = 0.$$

This ends the proof of this point.

- Case $b \geq a$. Applying the Leibniz integral rule with respect to a and using Proposition 2.1, the proof of which has been given simply, yields

$$\frac{\partial}{\partial a} I(a) = \frac{\partial}{\partial a}\left(\int_0^{+\infty} (1-\cos(ax))\cos(bx)\frac{1}{x^2}dx\right)$$
$$= \int_0^{+\infty} \frac{\partial}{\partial a}(1-\cos(ax))\cos(bx)\frac{1}{x^2}dx$$
$$= \int_0^{+\infty} -(-x\sin(ax))\cos(bx)\frac{1}{x^2}dx$$
$$= \int_0^{+\infty} \sin(ax)\cos(bx)\frac{1}{x}dx = 0.$$

Therefore, $I(a)$ is equal to a constant independent of a, like $I(b)$. Using the result of the first point, we derive

$$I(a) = I(b) = 0.$$

This ends the proof of this point.

This completes the proof. □

The key points of the proof were determining the integral at $a = b$ and using the Leibniz integral rule in conjunction with Proposition 2.1. Proposition 2.2 also shows that the oscillatory functions $(1-\cos(ax))/x^2$ and $\cos(bx)$ are, in a sense, orthogonal over $(0, +\infty)$. This orthogonality corresponds to an exact cancellation between the positive and negative contributions of the integrand over $(0, +\infty)$.

By combining Propositions 2.1 and 2.2, we obtain the following general integral formula:

$$\int_0^{+\infty} f(x)\cos(bx)\frac{1}{x}dx = 0,$$

where

$$f(x) \in \left\{\sin(ax), (1-\cos(ax))\frac{1}{x}\right\}$$

for any $b > a$. This compact formula unifies two distinct cases under a single framework, and it may serve as a basis for further generalizations to broader classes of oscillatory integrals.

2.3. **Third formula.** The third formula is presented in the proposition below. It corresponds to [6, Entry 3.8683 with $a = a^2$ and $b = b^2$].

Proposition 2.3. *Let* $a, b > 0$. *Then we have*

$$\int_0^{+\infty} \sin\left(ax - \frac{b}{x}\right)\frac{1}{x}dx = 0.$$

Proof. A standard trigonometric formula gives

$$\sin\left(ax - \frac{b}{x}\right) = \sin(ax)\cos\left(\frac{b}{x}\right) - \cos(ax)\sin\left(\frac{b}{x}\right).$$

Using this and the convergence of the involved integrals, we get

$$\begin{aligned}
&\int_0^{+\infty} \sin\left(ax - \frac{b}{x}\right)\frac{1}{x}dx \\
&= \int_0^{+\infty} \left(\sin(ax)\cos\left(\frac{b}{x}\right) - \cos(ax)\sin\left(\frac{b}{x}\right)\right)\frac{1}{x}dx \\
&= \int_0^{+\infty} \sin(ax)\cos\left(\frac{b}{x}\right)\frac{1}{x}dx - \int_0^{+\infty} \cos(ax)\sin\left(\frac{b}{x}\right)\frac{1}{x}dx.
\end{aligned}$$

Making the change of variables $x = (b/a)(1/y)$, with $y > 0$ because $a, b > 0$ (this plays a role in the definition of the limits of integration), in the second integral, we obtain

$$\begin{aligned}
&\int_0^{+\infty} \sin(ax)\cos\left(\frac{b}{x}\right)\frac{1}{x}dx - \int_0^{+\infty} \cos(ax)\sin\left(\frac{b}{x}\right)\frac{1}{x}dx \\
&= \int_0^{+\infty} \sin(ax)\cos\left(\frac{b}{x}\right)\frac{1}{x}dx \\
&\quad - \int_{+\infty}^0 \cos\left(a\left(\frac{b}{a}\right)\frac{1}{y}\right)\sin\left(\frac{b}{(b/a)(1/y)}\right)\frac{1}{(b/a)(1/y)}\left(\frac{b}{a}\right)\left(-\frac{1}{y^2}\right)dy \\
&= \int_0^{+\infty} \sin(ax)\cos\left(\frac{b}{x}\right)\frac{1}{x}dx - \int_0^{+\infty} \sin(ax)\cos\left(\frac{b}{x}\right)\frac{1}{x}dx = 0.
\end{aligned}$$

This completes the proof. □

The proof does not involve the evaluation of any intermediate integrals; it primarily relies on trigonometric formulas and changes of variables. Moreover, Proposition 2.3 identifies a simple function f for which

$$\int_0^{+\infty} \sin(f(x)) \frac{1}{x} dx = 0.$$

In particular, the choice

$$f(x) = ax - \frac{b}{x},$$

with $a, b > 0$, is valid, which demonstrates the generality of this result and provides a wide class of examples.

3. Conclusion

In this paper, we provide complete proofs of three two-parameter trigonometric integrals referenced in the literature that evaluate to zero, highlighting their non-trivial oscillatory structure. The employed methods rely solely on trigonometric formulas and standard integration techniques, thus avoiding the need for intermediate integral evaluations. Future work could involve extending these methods to multi-parameter integrals, generalizing them to other classes of oscillatory integral and exploring their potential applications in analysis and mathematical physics.

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