ON A CHEBYSHEV-TYPE INTEGRAL INEQUALITY FOR HÖLDER-CONTINUOUS FUNCTIONS

Christophe Chesneau

ABSTRACT. This article primarily establishes a Chebyshev-type integral inequality for Hölder-continuous functions, an area that appears to have been understudied. Furthermore, we present a proposition that demonstrates how the main theorem can be applied under alternative sets of assumptions.

1. INTRODUCTION

The Chebyshev integral inequality is a well-known tool for relating the integrals of functions and their products, usually under the monotonicity assumption. A formal statement of this inequality is presented below. Let $a, b \in \mathbb{R}$ with $b > a$ and $f, g : [a, b] \to \mathbb{R}$ be two differentiable functions such that

$$(1.1) \qquad \sup_{x \in [a,b]} |f'(x)| < +\infty, \qquad \sup_{x \in [a,b]} |g'(x)| < +\infty.$$

Then we have

$$(1.2) \qquad \left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right|$$
$$\leq \frac{\eta \iota (b-a)^2}{12},$$

where

$$\eta = \sup_{x \in [a,b]} |f'(x)|, \qquad \iota = \sup_{x \in [a,b]} |g'(x)|.$$

We refer the reader to the book [9]. Modern developments concerning Chebyshev-type integral inequalities can be found in [1–8, 10]. However, to the best of the knowledge of the author, a general statement of a Chebyshev-type integral inequality for Hölder-continuous functions has not yet appeared in the literature. This seems to be an understudied topic.

The aim of this article is to address this issue by providing a concise and straightforward proof that makes use of the controlled regularity of Hölder-continuous functions. Our result is a natural extension of the classical Chebyshev integral inequality and could be a useful analytical tool in contexts where Hölder continuity is fundamental. Additionally, we present a proposition that illustrates how the main theorem can be applied under alternative sets of assumptions.

The remainder of this article is organized as follows: Section 2 contains the main results, while Section 3 provides concluding remarks and outlines several directions for future research.

2. RESULTS

2.1. **Main result.** Our main theorem is stated below, emphasizing the case of Hölder-continuous functions f and g.

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $b > a$ and $f, g : [a, b] \to \mathbb{R}$ be two functions such that there exist $\alpha, \beta, \kappa, \omega \geq 0$ satisfying, for any $x, y \in [a, b]$,*

(2.1) $\qquad |f(x) - f(y)| \leq \kappa |x - y|^{\alpha}, \qquad |g(x) - g(y)| \leq \omega |x - y|^{\beta}.$

Then we have

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right|$$
$$\leq \frac{\kappa \omega (b-a)^{\alpha+\beta}}{(\alpha+\beta+1)(\alpha+\beta+2)}.$$

Proof. By an integral decomposition, we can write

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right)$$

$$= \frac{1}{2(b-a)^2} \left((b-a) \int_a^b f(x)g(x)dx + (b-a) \int_a^b f(y)g(y)dx \right.$$

$$\left. - \left(\int_a^b f(x)dx \right) \left(\int_a^b g(y)dy \right) - \left(\int_a^b f(y)dy \right) \left(\int_a^b g(x)dx \right) \right)$$

$$= \frac{1}{2(b-a)^2} \int_a^b \int_a^b \left(f(x) - f(y) \right) \left(g(x) - g(y) \right) dxdy.$$

This and the triangle inequality give

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right|$$

$$(2.2) \quad = \frac{1}{2(b-a)^2} \left| \int_a^b \int_a^b \left(f(x) - f(y) \right) \left(g(x) - g(y) \right) dxdy \right|$$

$$\leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b |f(x) - f(y)| \, |g(x) - g(y)| \, dxdy.$$

Using the assumptions in Equation (2.1), we get

$$\int_a^b \int_a^b |f(x) - f(y)| \, |g(x) - g(y)| \, dxdy \leq \kappa\omega \int_a^b \int_a^b |x - y|^\alpha |x - y|^\beta dxdy$$

$$(2.3)$$

$$= \kappa\omega \int_a^b \int_a^b |x - y|^{\alpha+\beta} dxdy.$$

Let us determine this last double integral with standard decompositions and primitives. We have

$$\int_a^b \int_a^b |x - y|^{\alpha+\beta} dxdy = \int_a^b \left(\int_a^y |x - y|^{\alpha+\beta} dx + \int_y^b |x - y|^{\alpha+\beta} dx \right) dy$$

$$= \int_a^b \left(\int_a^y (y - x)^{\alpha+\beta} dx + \int_y^b (x - y)^{\alpha+\beta} dx \right) dy$$

$$= \int_a^b \left(\left[-\frac{(y - x)^{\alpha+\beta+1}}{\alpha + \beta + 1} \right]_{x=a}^{x=y} + \left[\frac{(x - y)^{\alpha+\beta+1}}{\alpha + \beta + 1} \right]_{x=y}^{x=b} \right) dy$$

$$= \frac{1}{\alpha + \beta + 1} \int_a^b \left((y-a)^{\alpha+\beta+1} + (b-y)^{\alpha+\beta+1} \right) dy$$

$$= \frac{1}{\alpha + \beta + 1} \left(\int_a^b (y-a)^{\alpha+\beta+1} dy + \int_a^b (b-y)^{\alpha+\beta+1} dy \right)$$

(2.4)
$$= \frac{1}{\alpha + \beta + 1} \left(\left[\frac{(y-a)^{\alpha+\beta+2}}{\alpha+\beta+2} \right]_{y=a}^{y=b} + \left[-\frac{(b-y)^{\alpha+\beta+2}}{\alpha+\beta+2} \right]_{y=a}^{y=b} \right)$$

$$= \frac{1}{\alpha + \beta + 1} \left(\frac{(b-a)^{\alpha+\beta+2}}{\alpha+\beta+2} + \frac{(b-a)^{\alpha+\beta+2}}{\alpha+\beta+2} \right)$$

$$= \frac{2(b-a)^{\alpha+\beta+2}}{(\alpha+\beta+1)(\alpha+\beta+2)}.$$

Combining Equations (2.2), (2.3) and (2.4), we obtain

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right|$$

$$\leq \frac{1}{(b-a)^2} \times \frac{1}{2} \times \frac{2(b-a)^{\alpha+\beta+2}}{(\alpha+\beta+1)(\alpha+\beta+2)}$$

$$= \frac{\kappa\omega(b-a)^{\alpha+\beta}}{(\alpha+\beta+1)(\alpha+\beta+2)}.$$

This completes the proof.   □

Let us now establish a connection between Theorem 2.1 and the classical Chebyshev integral inequality given in Equation (1.2). In the case where $f, g : [a,b] \to \mathbb{R}$ are two differentiable functions such that

$$\sup_{x \in [a,b]} |f'(x)| < +\infty, \qquad \sup_{x \in [a,b]} |g'(x)| < +\infty$$

corresponding to Equation (2.1), f and g are Lipschitz-continuous functions, i.e., Hölder-continuous functions with parameters $\alpha = 1$ and $\beta = 1$, respectively, and

$$\kappa = \sup_{x \in [a,b]} |f'(x)| = \eta, \qquad \omega = \sup_{x \in [a,b]} |g'(x)| = \iota.$$

Consequently, the upper bound in Theorem 2.1 reduces to

$$\frac{\kappa\omega(b-a)^{\alpha+\beta}}{(\alpha+\beta+1)(\alpha+\beta+2)} = \frac{\eta\iota(b-a)^2}{12},$$

which coincides with the bound in the classical Chebyshev integral inequality.

2.2. **A proposition.** In the proposition below, we establish a connection between integral conditions on the derivatives of f and g and the setting of Theorem 2.1.

Proposition 2.1. *Let $a, b \in \mathbb{R}$ with $b > a$, $p, r > 1$, $q = p/(p-1)$, $s = r/(r-1)$, and $f, g : [a, b] \to \mathbb{R}$ be two differentiable functions such that*

$$(2.5) \qquad \int_a^b |f'(x)|^p dx < +\infty, \qquad \int_a^b |g'(x)|^r dx < +\infty.$$

Then we have

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right|$$
$$\leq \frac{\gamma\theta(b-a)^{1/q+1/s}}{(1/q+1/s+1)(1/q+1/s+2)},$$

where

$$\gamma = \left(\int_a^b |f'(x)|^p dx \right)^{1/p}, \qquad \theta = \left(\int_a^b |g'(x)|^r dx \right)^{1/r}.$$

Proof. The proof consists of applying Theorem 2.1 to a suitable configuration. Applying the Hölder integral inequality to f at the parameters p and q and using Equation (2.5), we have

$$|f(x) - f(y)| = \left| \int_x^y f'(t)dt \right| \leq \left| \int_x^y |f'(t)|^p dt \right|^{1/p} \left| \int_x^y dx \right|^{1/q}$$
$$\leq \left(\int_a^b |f'(x)|^p dx \right)^{1/p} |x-y|^{1/q} = \gamma |x-y|^{1/q}.$$

Therefore, f is Hölder-continuous with the constant factor γ and the exponent $1/q$.

Similarly, applying the Hölder integral inequality to g at the parameters r and s and using Equation (2.5), we have

$$|g(x) - g(y)| = \left| \int_x^y g'(t)dt \right| \leq \left| \int_x^y |g'(t)|^r dt \right|^{1/r} \left| \int_x^y dx \right|^{1/s}$$
$$\leq \left(\int_a^b |g'(x)|^r dx \right)^{1/r} |x-y|^{1/s} = \theta |x-y|^{1/s}.$$

Therefore, g is Hölder-continuous with the constant factor θ and the exponent $1/s$. Applying Theorem 2.1 to f and g with the appropriate notation, so $\kappa = \gamma$, $\omega = \theta$, $\alpha = 1/q$ and $\beta = 1/s$, we have

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right|$$
$$\leq \frac{\gamma\theta(b-a)^{1/q+1/s}}{(1/q+1/s+1)(1/q+1/s+2)}.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

This proposition primarily illustrates how Theorem 2.1 can be applied in a setting where integral conditions on f and g ensure their Hölder continuity. In a sense, this completes the results in [8, 10].

3. Conclusion

In this article, we have filled a theoretical gap by establishing a general form of a Chebyshev-type integral inequality for Hölder-continuous functions. We have further applied this result to functions subject to integral constraints, thereby extending the scope of classical Chebyshev-type integral inequalities.

Future research may explore several directions, including the extension of these inequalities to functions defined on higher-dimensional domains, the incorporation of fractional or variable-order Hölder conditions, and potential applications to numerical integration and approximation theory. Such developments could further clarify the role of Hölder regularity in the broader theory of functional inequalities.

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DEPARTMENT OF MATHEMATICS

UNIVERSITY OF CAEN-NORMANDIE

UFR DES SCIENCES - CAMPUS 2, CAEN

FRANCE.

Email address: christophe.chesneau@gmail.com