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BOUNDS FOR A SPECIFIC CLASS OF COMPOSITE INTEGRALS

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ABSTRACT. In the spirit of the classical Steffensen inequality, we derive new lower and upper bounds for a class of composite integrals that depend on two functions. Our results are based on monotonicity and concavity assumptions for only one of the functions. These assumptions are used alongside basic techniques such as changing the variables, monotonicity and concavity to prove our results. These techniques yield several distinct inequalities with a wide range of applications in analysis.

1. INTRODUCTION

Integral inequalities occupy a central position in mathematics. They are essential for obtaining precise estimates in many contexts, including functional equations, variational problems and differential systems. Well-known examples include the Cauchy-Schwarz, Hardy, Hilbert, Hölder, Minkowski, Steffensen, Wirtinger, and Young integral inequalities, which form the basis of many results in analysis and its applications. Comprehensive treatments of classical results and foundational techniques can be found in [1, 2, 8, 10, 20, 21]. In recent years, there has been a growing interest in refining and extending traditional integral inequalities. This has involved introducing new kernel structures and

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weight functions, as well as generalizing known results to broader classes of functions and functional spaces. Notable contributions in this area can be found in [3–7, 9, 11–14, 16–19].

The formulation of new integral inequalities strengthens theoretical foundations of the field and broadens its applicability across modern branches of analysis. In this article, we contribute to this ongoing development by determining lower and upper bounds for the following composite integral:

$$\int_0^{\int_0^1 g(t)dt} f(x)dx,$$

where $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to [0,1]$ are integrable functions. Inequalities of this form are in the spirit of the classical Steffensen integral inequality (see [15]). Our approach yields new results by imposing monotonicity or concavity assumptions on f alone. Several distinct lower and upper bounds are derived and supported by direct and elementary proofs based on fundamental techniques such as the change of variables, monotonicity and concavity. We also discuss possible applications of these results.

The remainder of the article is organized as follows: Section 2 presents the main theorem together with its counterpart and detailed proofs. Concluding remarks are given in Section 3.

2. Results

2.1. **Main theorem.** The statement of our main theorem is provided below, focusing on lower bounds of the main composite integral under various assumptions.

Theorem 2.1. *Let $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to [0,1]$ be two integrable functions.*

(1) *If f is decreasing, then we have*

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \geq \left(\int_0^1 g(t)dt\right)\left(\int_0^1 f(x)dx\right).$$

(2) *Alternatively, if f is decreasing, then we have*

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \geq \left(\int_0^1 g(t)dt\right) f\left(\int_0^1 g(t)dt\right).$$

(3) *If f is concave, then we have*

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \geq \left(\int_0^1 g(t)dt\right) \int_0^1 \int_0^1 f\left(xg(t)\right) dtdx.$$

(4) *If f is concave and $f(0) = 0$, then we have*

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \geq \frac{1}{2} \left(\int_0^1 g(t)dt\right) f\left(\int_0^1 g(t)dt\right).$$

(5) *Alternatively, if f is concave and $f(0) = 0$, then we have*

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \geq \left(\int_0^1 g(t)dt\right)^2 \left(\int_0^1 f(x)dx\right).$$

Proof. We begin by rewriting the main integral in a more convenient form. Using the change of variables $x = y\left(\int_0^1 g(t)dt\right)$ and standardizing the notation, we obtain

$$\int_0^{\int_0^1 g(t)dt} f(x)dx = \int_0^1 f\left(y\left(\int_0^1 g(t)dt\right)\right)\left(\left(\int_0^1 g(t)dt\right) dy\right)$$

$$= \left(\int_0^1 g(t)dt\right) \int_0^1 f\left(x\left(\int_0^1 g(t)dt\right)\right) dx. \tag{2.1}$$

This equivalent representation expresses the integral over the fixed interval $[0, 1]$, which simplifies the subsequent derivation of inequalities. We shall then prove each of the stated results separately, based on this formulation.

(1) Let us suppose that f is decreasing. Since $\int_0^1 g(t)dt \in [0, 1]$ because $g(t) \in [0, 1]$ for any $t \in [0, 1]$, we have $x\left(\int_0^1 g(t)dt\right) \leq x$ for any $x \in [0, 1]$. Since f is decreasing, we derive

$$f\left(x\left(\int_0^1 g(t)dt\right)\right) \geq f(x). \tag{2.2}$$

Combining Equations (2.1) and (2.2), we get

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \geq \left(\int_0^1 g(t)dt\right)\left(\int_0^1 f(x)dx\right).$$

The inequality is thus established.

(2) Let us suppose that f is decreasing. We have $x\left(\int_0^1 g(t)dt\right) \leq \int_0^1 g(t)dt$ for any $x \in [0,1]$. Since f is decreasing, we obtain

$$f\left(x\left(\int_0^1 g(t)dt\right)\right) \geq f\left(\int_0^1 g(t)dt\right). \tag{2.3}$$

Combining Equations (2.1) and (2.3), we get

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \geq \left(\int_0^1 g(t)dt\right) f\left(\int_0^1 g(t)dt\right).$$

The desired inequality is obtained.

(3) Let us suppose that f is concave. Then the Jensen integral inequality applied to f and the uniform measure on $[0,1]$ gives

$$f\left(x\left(\int_0^1 g(t)dt\right)\right) = f\left(\int_0^1 xg(t)dt\right) \geq \int_0^1 f\left(xg(t)\right) dt. \tag{2.4}$$

Combining Equations (2.1) and (2.4), we get

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \geq \left(\int_0^1 g(t)dt\right) \int_0^1 \int_0^1 f\left(xg(t)\right) dtdx.$$

The inequality is thus established.

(4) Let us suppose that f is concave and $f(0) = 0$. Then we have

$$f\left(x\left(\int_0^1 g(t)dt\right)\right) = f\left(x\left(\int_0^1 g(t)dt\right) + (1-x) \times 0\right)$$

$$\geq xf\left(\int_0^1 g(t)dt\right) + (1-x)f(0) = xf\left(\int_0^1 g(t)dt\right). \tag{2.5}$$

Combining Equations (2.1) and (2.5), we get

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \geq \left(\int_0^1 g(t)dt\right) f\left(\int_0^1 g(t)dt\right) \left(\int_0^1 xdx\right)$$

$$= \frac{1}{2}\left(\int_0^1 g(t)dt\right) f\left(\int_0^1 g(t)dt\right).$$

The desired inequality is obtained.

(5) Let us suppose that f is concave and $f(0) = 0$. Since $\int_0^1 g(t)dt \in [0,1]$ because $g(t) \in [0,1]$ for any $t \in [0,1]$, we have

$$f\left(x\left(\int_0^1 g(t)dt\right)\right) = f\left(\left(\int_0^1 g(t)dt\right)x + \left(1 - \int_0^1 g(t)dt\right) \times 0\right)$$

$$\geq \left(\int_0^1 g(t)dt\right) f(x) + \left(1 - \int_0^1 g(t)dt\right) f(0) = \left(\int_0^1 g(t)dt\right) f(x). \tag{2.6}$$

Combining Equations (2.1) and (2.6), we get

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \geq \left(\int_0^1 g(t)dt\right)\left(\int_0^1 g(t)dt\right)\left(\int_0^1 f(x)dx\right)$$

$$= \left(\int_0^1 g(t)dt\right)^2 \left(\int_0^1 f(x)dx\right).$$

The inequality is thus established.

This completes the proof. □

We can observe that the bound in item 3 involves a double integral, which is relatively complex. This departs from the existing theory on the Steffensen integral inequality, among others.

Theorems such as Theorem 2.1 are particularly relevant because inequalities involving composite integrals can be used to derive sharp estimates and comparison results in functional and integral analysis, as well as in the study of variational and differential problems.

2.2. **Counterpart.** The counterpart of Theorem 2.1 is given below, where increasing and convex functions are considered instead of decreasing and concave functions. This has the effect of reversing the inequalities. Upper bounds of the main composite integral are thus obtained.

Theorem 2.2. *Let* $f : [0, 1] \to \mathbb{R}$ *and* $g : [0, 1] \to [0, 1]$ *be two integrable functions.*

(1) *If* f *is increasing, then we have*

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \leq \left(\int_0^1 g(t)dt\right)\left(\int_0^1 f(x)dx\right).$$

(2) *Alternatively, if* f *is increasing, then we have*

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \leq \left(\int_0^1 g(t)dt\right) f\left(\int_0^1 g(t)dt\right).$$

(3) *If f is convex, then we have*

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \leq \left(\int_0^1 g(t)dt\right) \int_0^1 \int_0^1 f\left(xg(t)\right) dtdx.$$

(4) *If f is convex and $f(0) = 0$, then we have*

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \leq \frac{1}{2}\left(\int_0^1 g(t)dt\right) f\left(\int_0^1 g(t)dt\right).$$

(5) *Alternatively, if f is convex and $f(0) = 0$, then we have*

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \leq \left(\int_0^1 g(t)dt\right)^2 \left(\int_0^1 f(x)dx\right).$$

Proof. We follow the approach in Theorem 2.1. Using the change of variables $x = y\left(\int_0^1 g(t)dt\right)$ and standardizing the notation, we get

$$\int_0^{\int_0^1 g(t)dt} f(x)dx = \int_0^1 f\left(y\left(\int_0^1 g(t)dt\right)\right)\left(\left(\int_0^1 g(t)dt\right)dy\right)$$

$$= \left(\int_0^1 g(t)dt\right)\int_0^1 f\left(x\left(\int_0^1 g(t)dt\right)\right)dx. \tag{2.7}$$

We shall prove each of the stated results separately based on this formulation.

(1) Let us suppose that f is increasing. Since $\int_0^1 g(t)dt \in [0,1]$ because $g(t) \in [0,1]$ for any $t \in [0,1]$, we have $x\left(\int_0^1 g(t)dt\right) \leq x$ for any $x \in [0,1]$. Since f is increasing, we derive

$$f\left(x\left(\int_0^1 g(t)dt\right)\right) \leq f(x). \tag{2.8}$$

Combining Equations (2.7) and (2.8), we get

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \leq \left(\int_0^1 g(t)dt\right)\left(\int_0^1 f(x)dx\right).$$

The desired inequality is obtained.

(2) Let us suppose that f is increasing. Since, $x\left(\int_0^1 g(t)dt\right) \leq \int_0^1 g(t)dt$ for any $x \in [0,1]$, we obtain

$$f\left(x\left(\int_0^1 g(t)dt\right)\right) \leq f\left(\int_0^1 g(t)dt\right). \tag{2.9}$$

Combining Equations (2.7) and (2.9), we get

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \leq \left(\int_0^1 g(t)dt\right) f\left(\int_0^1 g(t)dt\right).$$

The inequality is thus established.

(3) Let us suppose that f is convex. Then the Jensen integral inequality applied to f and the uniform measure on $[0,1]$ gives

$$f\left(x\left(\int_0^1 g(t)dt\right)\right) = f\left(\int_0^1 xg(t)dt\right) \leq \int_0^1 f\left(xg(t)\right)dt. \tag{2.10}$$

Combining Equations (2.7) and (2.10), we get

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \leq \left(\int_0^1 g(t)dt\right) \int_0^1 \int_0^1 f\left(xg(t)\right)dtdx.$$

The desired inequality is obtained.

(4) Let us suppose that f is convex and $f(0) = 0$. Then we have

$$f\left(x\left(\int_0^1 g(t)dt\right)\right) = f\left(x\left(\int_0^1 g(t)dt\right) + (1-x)\times 0\right)$$
$$\leq xf\left(\int_0^1 g(t)dt\right) + (1-x)f(0) = xf\left(\int_0^1 g(t)dt\right). \tag{2.11}$$

Combining Equations (2.7) and (2.11), we get

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \leq \left(\int_0^1 g(t)dt\right) f\left(\int_0^1 g(t)dt\right)\left(\int_0^1 xdx\right)$$
$$= \frac{1}{2}\left(\int_0^1 g(t)dt\right) f\left(\int_0^1 g(t)dt\right).$$

The inequality is thus established.

(5) Let us suppose that f is convex and $f(0) = 0$. Since $\int_0^1 g(t)dt \in [0,1]$ because $g(t) \in [0,1]$ for any $t \in [0,1]$, we have

$$f\left(x\left(\int_0^1 g(t)dt\right)\right) = f\left(\left(\int_0^1 g(t)dt\right)x + \left(1 - \int_0^1 g(t)dt\right)\times 0\right)$$
$$\leq \left(\int_0^1 g(t)dt\right) f(x) + \left(1 - \int_0^1 g(t)dt\right) f(0) = \left(\int_0^1 g(t)dt\right) f(x). \tag{2.12}$$

Combining Equations (2.7) and (2.12), we get

$$\int_0^{\int_0^1 g(t)dt} f(x)dx \leq \left(\int_0^1 g(t)dt\right)\left(\int_0^1 g(t)dt\right)\left(\int_0^1 f(x)dx\right)$$
$$= \left(\int_0^1 g(t)dt\right)^2 \left(\int_0^1 f(x)dx\right).$$

The desired inequality is obtained.

This completes the proof. □

The potential applications of this theorem are the same as those of Theorem 2.1, except that they deal with different assumptions on f.

3. Conclusion

In this note, we established new lower and upper bounds for composite integrals of the form

$$\int_0^{\int_0^1 g(t)dt} f(x)dx,$$

relying only on monotonicity or concavity assumptions on f. The results provide flexible and elementary tools for deriving integral inequalities of the Steffensen type and extend the classical theory of integral inequalities. Future research may explore multidimensional extensions, sharper bounds, and applications to weighted inequalities in differential equations, functional analysis, and variational problems.

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