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CONTRIBUTIONS TO CONVEX INTEGRAL INEQUALITIES

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ABSTRACT. This article advances research on convex integral inequalities through two new theorems that integrate key properties of convex functions. The results are presented with full proofs to ensure rigor and clarity. Numerical examples illustrate both the practical relevance and the effectiveness of the proposed inequalities.

1. INTRODUCTION

Among the most fundamental concepts in mathematics are convex functions. Their formal definition is given below. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$ be such that $a < b$. A function $f : [a, b] \to \mathbb{R}$ is called convex on $[a, b]$ if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, the following inequality is satisfied:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \tag{1.1}$$

Two important properties of convex functions are listed below:

Key property 1: If f is twice differentiable and convex on $[a, b]$, then f' is non-decreasing.

Key property 2: If f is twice differentiable, convex on $[a, b]$ with $a \geq 0$, and satisfies $f(0) = 0$, then the ratio $f(x)/x$ is non-decreasing on $[a, b]$.

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The concept of convexity has many significant implications, including a wide range of integral inequalities commonly referred to as 'convex integral inequalities'. Prominent examples include the Jensen and Hermite-Hadamard integral inequalities, which reveal fundamental links between the values of a convex function and its averages over an interval. These inequalities have inspired numerous extensions and refinements in modern analysis and have many applications in mathematics. Further details can be found in [1–15].

Despite the substantial existing body of work on this topic, the search for new convex integral inequalities remains an active and popular area of research. These inequalities offer opportunities for new applications and more in-depth theoretical developments. This article makes contributions to these efforts. More precisely, we present two new theorems, each of which establishes new convex integral inequalities. Both theorems share the common feature of combining the two key properties of convex functions discussed above, yet remain relatively simple and accessible. Full details of the proofs are provided to ensure clarity and rigor. Several numerical examples are provided to illustrate the theory.

The rest of the article is as follows: The two main theorems are presented in Section 2. A conclusion is given in Section 3.

2. Results

2.1. **First theorem.** The theorem below establishes our first convex integral inequalities, based on the key properties of convex functions.

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $b > a \geq 0$, and $\varphi, \phi : [a, b] \to \mathbb{R}$ be two twice differentiable convex functions such that $\varphi(0) = 0$. Then the following inequalities holds:*

$$\int_a^b \frac{x}{\varphi(x)} \phi'(x) dx \leq \frac{\phi(b) - \phi(a)}{b - a} \left(\int_a^b \frac{x}{\varphi(x)} dx \right)$$

and

$$\int_a^b \frac{\varphi(x)}{x} \phi'(x) dx \geq \frac{\phi(b) - \phi(a)}{b - a} \left(\int_a^b \frac{\varphi(x)}{x} dx \right),$$

provided that the integrals exist.

Proof. The proof is based on the key properties of convex functions, which are detailed below. Since ϕ is a twice differentiable convex function, ϕ' is non-decreasing. Moreover, since φ is a twice differentiable convex function such that $\varphi(0) = 0$, $\varphi(x)/x$ is non-decreasing. It follows from the Chebyshev integral inequality applied to the functions $x/\varphi(x)$ and $\phi'(x)$ with different monotonicity that

$$\int_a^b \frac{x}{\varphi(x)}\phi'(x)dx \leq \frac{1}{b-a}\left(\int_a^b \frac{x}{\varphi(x)}dx\right)\left(\int_a^b \phi'(x)dx\right)$$
$$= \frac{\phi(b)-\phi(a)}{b-a}\left(\int_a^b \frac{x}{\varphi(x)}dx\right).$$

Moreover, it follows from the Chebyshev integral inequality applied to the functions $\varphi(x)/x$ and $\phi'(x)$ with the same monotonicity that

$$\int_a^b \frac{\varphi(x)}{x}\phi'(x)dx \geq \frac{1}{b-a}\left(\int_a^b \frac{\varphi(x)}{x}dx\right)\left(\int_a^b \phi'(x)dx\right)$$
$$= \frac{\phi(b)-\phi(a)}{b-a}\left(\int_a^b \frac{\varphi(x)}{x}dx\right).$$

This completes the proof of the theorem. □

Two numerical examples of this theorem are detailed below.

Example 1. *We consider* $a = 0$, $b = 1$, $\varphi(x) = e^x - 1$ *and* $\phi(x) = x^2$. *Then the required assumptions are satisfied, and we have*

$$\int_a^b \frac{x}{\varphi(x)}\phi'(x)dx = 2\int_0^1 \frac{x^2}{e^x-1}dx \approx 0.707878$$

and

$$\frac{\phi(b)-\phi(a)}{b-a}\left(\int_a^b \frac{x}{\varphi(x)}dx\right) = \int_0^1 \frac{x}{e^x-1}dx \approx 0.777505.$$

We have $0.707878 < 0.777505$, *supporting the first inequality in Theorem 2.1.*

To illustrate the second inequality, we compute

$$\int_a^b \frac{\varphi(x)}{x}\phi'(x)dx = 2\int_0^1 (e^x-1)dx \approx 1.4366$$

and

$$\frac{\phi(b)-\phi(a)}{b-a}\left(\int_a^b \frac{\varphi(x)}{x}dx\right) = \int_0^1 \frac{e^x-1}{x}dx \approx 1.3179.$$

We have $1.3179 < 1.4366$*, supporting the second inequality in Theorem 2.1.*

Example 2. *We consider* $a = 0$*,* $b = 1$*,* $\varphi(x) = \sinh(x)$ *and* $\phi(x) = e^x$*. Then the required assumptions are satisfied, and we have*

$$\int_a^b \frac{x}{\varphi(x)}\phi'(x)dx = \int_0^1 \frac{x}{\sinh(x)}e^x dx \approx 1.60695$$

and

$$\frac{\phi(b)-\phi(a)}{b-a}\left(\int_a^b \frac{x}{\varphi(x)}dx\right) = (e-1)\int_0^1 \frac{x}{\sinh(x)}dx \approx 1.629037.$$

We have $1.60695 < 1.629037$*, supporting the first inequality in Theorem 2.1.*

To illustrate the second inequality, we compute

$$\int_a^b \frac{\varphi(x)}{x}\phi'(x)dx = \int_0^1 \frac{\sinh(x)}{x}e^x dx \approx 1.84194$$

and

$$\frac{\phi(b)-\phi(a)}{b-a}\left(\int_a^b \frac{\varphi(x)}{x}dx\right) = (e-1)\int_0^1 \frac{\sinh(x)}{x}dx \approx 1.81665.$$

We have $1.81665 < 1.84194$*, supporting the second inequality in Theorem 2.1.*

2.2. **Second theorem.**

The theorem below establishes our second convex integral inequalities, based on the key properties of convex functions.

Theorem 2.2. *Let* $a, b \in \mathbb{R}$ *with* $b > a \geq 0$*, and* $\varphi, \phi : [a, b] \to [0, +\infty)$ *be two twice differentiable convex functions such that* $\varphi(0) = 0$ *and* $\phi(0) = 0$*. Then the following inequality holds:*

$$\left(\int_a^b \frac{\varphi(x)}{\phi(x)}dx\right)\left(\int_a^b \frac{\varphi(x)\phi(x)}{x^2}dx\right)^{-1} \leq \left(\int_a^b \frac{x}{\phi(x)}dx\right)\left(\int_a^b \frac{\phi(x)}{x}dx\right)^{-1}$$

or, equivalently,

$$\left(\int_a^b \frac{\varphi(x)}{\phi(x)}dx\right)\left(\int_a^b \frac{\phi(x)}{x}dx\right) \leq \left(\int_a^b \frac{x}{\phi(x)}dx\right)\left(\int_a^b \frac{\varphi(x)\phi(x)}{x^2}dx\right),$$

provided that the integrals exist.

Proof. The proof is based on the key properties of convex functions, which are detailed below. Since φ and ϕ are two twice differentiable convex functions such that $\varphi(0) = 0$ and $\phi(0) = 0$, $\varphi(x)/x$ and $\phi(x)/x$ are non-decreasing. It follows from the Chebyshev integral inequality applied to the functions $\varphi(x)/x$ and $x/\phi(x)$ with different monotonicity that

$$\int_a^b \frac{\varphi(x)}{\phi(x)}dx = \int_a^b \frac{\varphi(x)}{x} \times \frac{x}{\phi(x)}dx \leq \frac{1}{b-a}\left(\int_a^b \frac{\varphi(x)}{x}dx\right)\left(\int_a^b \frac{x}{\phi(x)}dx\right). \tag{2.1}$$

Moreover, it follows from the Chebyshev integral inequality applied to the functions $\varphi(x)/x$ and $\phi(x)/x$ with the same monotonicity that

$$\int_a^b \frac{\varphi(x)\phi(x)}{x^2}dx = \int_a^b \frac{\varphi(x)}{x} \times \frac{\phi(x)}{x}dx \geq \frac{1}{b-a}\left(\int_a^b \frac{\varphi(x)}{x}dx\right)\left(\int_a^b \frac{\phi(x)}{x}dx\right).$$

This implies that

$$\left(\int_a^b \frac{\varphi(x)\phi(x)}{x^2}dx\right)^{-1} \leq (b-a)\left(\int_a^b \frac{\varphi(x)}{x}dx\right)^{-1}\left(\int_a^b \frac{\phi(x)}{x}dx\right)^{-1}. \tag{2.2}$$

Since the integrals are non-negative because of the non-negativity of φ and ϕ, multiplying Equations (2.1) and (2.2), we get

$$\left(\int_a^b \frac{\varphi(x)}{\phi(x)}dx\right)\left(\int_a^b \frac{\varphi(x)\phi(x)}{x^2}dx\right)^{-1} \leq \left(\int_a^b \frac{x}{\phi(x)}dx\right)\left(\int_a^b \frac{\phi(x)}{x}dx\right)^{-1}.$$

This can also be written as

$$\left(\int_a^b \frac{\varphi(x)}{\phi(x)}dx\right)\left(\int_a^b \frac{\phi(x)}{x}dx\right) \leq \left(\int_a^b \frac{x}{\phi(x)}dx\right)\left(\int_a^b \frac{\varphi(x)\phi(x)}{x^2}dx\right).$$

This completes the proof. □

Two numerical examples of this theorem are detailed below.

Example 3. *We consider $a = 0$, $b = 1$, $\varphi(x) = e^x - 1$ and $\phi(x) = \sinh(x)$. Then the required assumptions are satisfied, and we have*

$$\left(\int_a^b \frac{\varphi(x)}{\phi(x)}dx\right)\left(\int_a^b \frac{\varphi(x)\phi(x)}{x^2}dx\right)^{-1}$$
$$= \left(\int_0^1 \frac{e^x-1}{\sinh(x)}dx\right)\left(\int_0^1 \frac{(e^x-1)\sinh(x)}{x^2}dx\right)^{-1}$$
$$\approx 1.2402 \times 0.71230 \approx 0.88339446$$

and

$$\left(\int_a^b \frac{x}{\phi(x)}dx\right)\left(\int_a^b \frac{\phi(x)}{x}dx\right)^{-1}$$
$$= \left(\int_0^1 \frac{x}{\sinh(x)}dx\right)\left(\int_0^1 \frac{\sinh(x)}{x}dx\right)^{-1}$$
$$\approx 0.948062 \times 0.945849 \approx 0.8967234.$$

We have $0.88339446 < 0.8967234$*, supporting the first inequality in Theorem 2.2.*

Example 4. *We consider* $a = 0$*,* $b = 1$*,* $\varphi(x) = 1 - \cos(x)$ *and* $\phi(x) = e^x - 1$*. Then the required assumptions are satisfied, and we have*

$$\left(\int_a^b \frac{\varphi(x)}{\phi(x)}dx\right)\left(\int_a^b \frac{\varphi(x)\phi(x)}{x^2}dx\right)^{-1}$$
$$= \left(\int_0^1 \frac{1-\cos(x)}{e^x - 1}dx\right)\left(\int_0^1 \frac{(1-\cos(x))(e^x-1)}{x^2}dx\right)^{-1}$$
$$\approx 0.170293 \times 2.9115146 \approx 0.49581055$$

and

$$\left(\int_a^b \frac{x}{\phi(x)}dx\right)\left(\int_a^b \frac{\phi(x)}{x}dx\right)^{-1}$$
$$= \left(\int_0^1 \frac{x}{e^x - 1}dx\right)\left(\int_0^1 \frac{e^x - 1}{x}dx\right)^{-1}$$
$$\approx 0.777505 \times 0.758782 \approx 0.5899567.$$

We have $0.49581055 < 0.5899567$*, supporting the first inequality in Theorem 2.2.*

3. Conclusion

In conclusion, we have developed two new convex integral inequalities that capture the key features of convex functions in a simple yet robust framework. These results enrich the theoretical basis of convex analysis. They also pave the way for further exploration of related inequalities and their applications. Future research could involve extending these results to broader classes of functions, or investigating their implications in applied fields such as optimization and numerical analysis.

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