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A NOTE ON A SOPHISTICATED INTEGRAL INEQUALITY

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ABSTRACT. This note presents an extension of an integral inequality distinguished by its elegance and technical complexity. For the sake of completeness and clarity, the full proof is provided. Additionally, an application to the beta function is proposed.

1. INTRODUCTION

Integral inequalities play a fundamental role in mathematics. They underpin many results in real analysis, functional analysis, probability theory and applied mathematics. Detailed expositions of the classical integral results and the techniques underlying them can be found in [1, 2, 8, 10, 22, 23]. In recent decades, there has been an increasing focus on refining and generalizing these inequalities. Representative contributions reflecting this ongoing development can be found in [3–7, 9, 11, 15–21].

In a post by Misaki on the mathematical forum MathOverflow [14], the integral inequality stated in the theorem below is discussed.

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Theorem 1.1. *Let $f : [0,1] \to [0,\infty)$ be a continuously differentiable function satisfying $f(1) = 0$. Then we have*

$$4\int_0^1 x^2 f'(x)^2 dx \geq \int_0^1 f(x)^2 dx + \left(\int_0^1 f(x)dx\right)^2.$$

This result is elegant because it succinctly captures the interaction between a function and its derivative under natural boundary conditions. It also reflects structural features common to several classical results in analysis, such as the Hardy, Wirtinger and Poincaré integral inequalities. Two distinct and rather technical proofs were provided on MathOverflow. The first, Proof I by [12], relies on a double-integral representation involving the minimum and maximum of the integration variables. The second, Proof II by [13], employs auxiliary (intermediate) functions together with the Cauchy-Schwarz integral inequality.

In the present note, we extend this integral inequality by introducing a tunable parameter $p \in [0,1)$. The proof follows the same general strategy as Proof II, adapting its key ideas to accommodate the additional parameter. All derivations are included in full detail to ensure clarity and transparency. An application to the beta function is also provided.

The remainder of this note is organized as follows: In Section 2, we present our generalized integral inequality. Section 3 gives the conclusion and provides insights into new research.

2. Main result

The one-parameter extension of the inequality in Theorem 1.1 is presented in the theorem below, followed by its proof, a brief discussion and an application to the beta function.

Theorem 2.1. *Let $p \in [0,1)$ and $f : [0,1] \to [0,\infty)$ be a continuously differentiable function satisfying $f(1) = 0$. Then we have*

$$\int_0^1 x^{2p+1} f'(x)^2 dx \geq p^2 \int_0^1 x^{2p-1} f(x)^2 dx + 2(1-p)^3 \left(\int_0^1 f(x)dx\right)^2,$$

provided that the integrals involved exist.

Proof. Throughout the proof, we work with the transformed function

$$g(s) = f(e^{-s}).$$

A key step in the argument is to express the relevant integrals in terms of this function.

First, noticing that

$$g'(s) = -e^{-s} f'(e^{-s})$$

and changing the variables $x = e^{-s}$, $s \in [0, \infty)$, we get

$$\int_0^1 x^{2p+1} f'(x)^2 dx = \int_\infty^0 e^{-s(2p+1)} f'(e^{-s})^2 (-e^{-s}) ds$$

$$= \int_0^\infty \left(-e^{-s} f'(e^{-s})\right)^2 e^{-2sp} ds = \int_0^\infty g'(s)^2 e^{-2sp} ds. \tag{2.1}$$

We now define the intermediary function

$$v(s) = e^{-sp} g(s).$$

Then we have

$$v'(s) = e^{-sp} g'(s) - p e^{-sp} g(s) = e^{-sp} g'(s) - p v(s),$$

so that

$$g'(s) e^{-sp} = v'(s) + p v(s).$$

Therefore, we have

$$\int_0^\infty g'(s)^2 e^{-2sp} ds = \int_0^\infty \left(v'(s) + p v(s)\right)^2 ds$$

$$= \int_0^\infty \left(v'(s)^2 + p^2 v(s)^2 + 2p v(s) v'(s)\right) ds$$

$$= \int_0^\infty v'(s)^2 ds + p^2 \int_0^\infty v(s)^2 ds + 2p \int_0^\infty v'(s) v(s) ds.$$

Since $f(1) = 0$ and $f(0)$ exists, we compute the last integral as

$$\int_0^\infty v'(s) v(s) ds = \left[\frac{1}{2} v(s)^2\right]_0^\infty = \left[\frac{1}{2} e^{-2sp} f(e^{-s})^2\right]_0^\infty$$

$$= \frac{1}{2}\left[\lim_{s\to\infty} e^{-2sp} f(e^{-s})^2 - f(1)^2\right] = \frac{1}{2}\left[0 \times f(0)^2 - 0^2\right] = 0.$$

Thus, we have

$$\int_0^\infty g'(s)^2 e^{-2sp} ds = \int_0^\infty v'(s)^2 ds + p^2 \int_0^\infty v(s)^2 ds. \tag{2.2}$$

Changing back to the variables $x = e^{-s}$, $s \in [0, \infty)$, we can express the last integral as

$$\int_0^\infty v(s)^2 ds = \int_0^\infty e^{-2sp} g(s)^2 ds = \int_0^\infty e^{-2sp} f(e^{-s})^2 ds$$

$$= \int_1^0 x^{2p} f(x)^2 \left(-\frac{1}{x}\right) dx = \int_0^1 x^{2p-1} f(x)^2 dx. \tag{2.3}$$

Combining Equations (2.1), (2.2) and (2.3), we find that

$$\int_0^1 x^{2p+1} f'(x)^2 dx = \int_0^\infty v'(s)^2 ds + p^2 \int_0^1 x^{2p-1} f(x)^2 dx. \tag{2.4}$$

Therefore, to prove the desired inequality, it suffices to show that

$$\int_0^\infty v'(s)^2 ds \geq 2(1-p)^3 \Big(\int_0^1 f(x) dx \Big)^2.$$

We compute

$$\int_0^1 f(x) dx = \int_0^\infty g(s) e^{-s} ds = \int_0^\infty g(s) e^{-sp} e^{s(p-1)} ds = \int_0^\infty v(s) e^{s(p-1)} ds.$$

Integrating by parts, we can express this last integral as

$$\int_0^\infty v(s) e^{s(p-1)} ds = \left[\frac{1}{p-1} e^{s(p-1)} v(s)\right]_0^\infty + \frac{1}{1-p} \int_0^\infty v'(s) e^{s(p-1)} ds$$

$$= \left[\frac{1}{p-1} e^{s(p-1)} e^{-sp} f(e^{-s})\right]_0^\infty + \frac{1}{1-p} \int_0^\infty v'(s) e^{s(p-1)} ds$$

$$= \frac{1}{p-1} \left[\lim_{s\to\infty} e^{-s} f(e^{-s}) - f(1)\right] + \frac{1}{1-p} \int_0^\infty v'(s) e^{s(p-1)} ds$$

$$= \frac{1}{p-1} [0 \times f(0) - f(1)] + \frac{1}{1-p} \int_0^\infty v'(s) e^{s(p-1)} ds$$

$$= 0 + \frac{1}{1-p} \int_0^\infty v'(s) e^{s(p-1)} ds = \frac{1}{1-p} \int_0^\infty v'(s) e^{s(p-1)} ds.$$

Based on this, applying the Cauchy-Schwarz integral inequality and using a standard exponential primitive with $p \in [0, 1)$, we obtain

$$\begin{aligned}\left(\int_0^1 f(x)dx\right)^2 &= \frac{1}{(1-p)^2}\left(\int_0^\infty v'(s)e^{s(p-1)}ds\right)^2\\ &\leq \frac{1}{(1-p)^2}\left(\int_0^\infty v'(s)^2ds\right)\left(\int_0^\infty e^{2s(p-1)}ds\right)\\ &= \frac{1}{2(1-p)^3}\int_0^\infty v'(s)^2ds,\end{aligned}$$

so that

$$\int_0^\infty v'(s)^2ds \geq 2(1-p)^3\Big(\int_0^1 f(x)dx\Big)^2. \tag{2.5}$$

Combining Equations (2.4) and (2.5), we get

$$\int_0^1 x^{2p+1}f'(x)^2dx \geq p^2\int_0^1 x^{2p-1}f(x)^2dx + 2(1-p)^3\left(\int_0^1 f(x)dx\right)^2.$$

This completes the proof. □

Setting $p = 1/2$, we have $p^2 = 2(1-p)^3 = 1/4$, and Theorem 2.1 gives

$$\int_0^1 x^2f'(x)^2dx \geq \frac{1}{4}\int_0^1 f(x)^2dx + \frac{1}{4}\left(\int_0^1 f(x)dx\right)^2,$$

so that

$$4\int_0^1 x^2f'(x)^2dx \geq \int_0^1 f(x)^2dx + \left(\int_0^1 f(x)dx\right)^2,$$

which corresponds to Theorem 1.1. To the best of our knowledge, each intermediate value of p gives rise to a distinct and novel integral inequality. In particular,

- for $p = 1/4$, we have

$$\int_0^1 x^{3/2}f'(x)^2dx \geq \frac{1}{16}\int_0^1 x^{-1/2}f(x)^2dx + \frac{27}{32}\left(\int_0^1 f(x)dx\right)^2,$$

- for $p = 1/6$, we have

$$\int_0^1 x^{4/3}f'(x)^2dx \geq \frac{1}{36}\int_0^1 x^{-2/3}f(x)^2dx + \frac{125}{108}\left(\int_0^1 f(x)dx\right)^2,$$

- for $p = 2/3$, we have

$$\int_0^1 x^{7/3} f'(x)^2 dx \geq \frac{4}{9} \int_0^1 x^{1/3} f(x)^2 dx + \frac{2}{27} \left(\int_0^1 f(x) dx \right)^2.$$

An application of Theorem 2.1 involving the beta function is presented in the proposition below.

Proposition 2.1. *Let $p \in [0, 1)$ and $\alpha > 1/2$. Then we have*

$$\alpha^2 B(2(p+1), 2\alpha - 1) \geq p^2 B(2p, 2\alpha + 1) + \frac{2(1-p)^3}{(\alpha+1)^2},$$

where $B(a, b)$ denotes the beta function defined by $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$ with $a, b > 0$.

Proof. Applying Theorem 2.1 with the function $f(x) = (1-x)^\alpha$ satisfying the required assumptions, we get

$$\int_0^1 x^{2p+1} \left(\alpha (1-x)^{\alpha-1} \right)^2 dx$$
$$\geq p^2 \int_0^1 x^{2p-1} (1-x)^{2\alpha} dx + 2(1-p)^3 \left(\int_0^1 (1-x)^\alpha dx \right)^2,$$

which can be rewritten as

$$\alpha^2 \int_0^1 x^{2(p+1)-1} (1-x)^{2\alpha-1-1} dx$$
$$\geq p^2 \int_0^1 x^{2p-1} (1-x)^{2\alpha+1-1} dx + \frac{2(1-p)^3}{(\alpha+1)^2}.$$

In terms of the beta function, we obtain

$$\alpha^2 B(2(p+1), 2\alpha - 1) \geq p^2 B(2p, 2\alpha + 1) + \frac{2(1-p)^3}{(\alpha+1)^2}.$$

This concludes the proof. □

Let us illustrate numerically Theorem 2.1.

- For $p = 1/4$ and $\alpha = 1$, we have

$$\alpha^2 B(2(p+1), 2\alpha - 1) = 0.4 \geq 0.2776 \approx p^2 B(2p, 2\alpha + 1) + \frac{2(1-p)^3}{(\alpha+1)^2}.$$

- For $p = 1/6$ and $\alpha = 2$, we have

$$\alpha^2 B(2(p+1), 2\alpha - 1) \approx 0.2373 \geq 0.1731 \approx p^2 B(2p, 2\alpha + 1) + \frac{2(1-p)^3}{(\alpha+1)^2}.$$

- For $p = 2/3$ and $\alpha = 2/3$, we have

$$\alpha^2 B(2(p+1), 2\alpha - 1) \approx 0.8244 \geq 0.1444 \approx p^2 B(2p, 2\alpha + 1) + \frac{2(1-p)^3}{(\alpha+1)^2}.$$

The theory is thus supported by these numerical examples.

3. Conclusion

This note generalizes a technically sophisticated integral inequality by introducing an adjustable parameter $p \in [0, 1)$. A complete and transparent proof is provided. An application to the beta function is proposed. The introduction of the parameter p opens up new avenues for further investigation, such as identifying sharp constants and examining analogous inequalities in broader functional settings or higher-dimensional domains. The excluded case of $p = 1$ is also an area for future research.

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