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ON TWO ONE-PARAMETER CONVEX INTEGRAL INEQUALITIES

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ABSTRACT. Convex integral inequalities form a fundamental part of mathematics. In this article, we make a contribution to this classical field by presenting two new one-parameter convex integral inequalities. Several illustrative examples are provided to demonstrate the applicability of the results.

1. INTRODUCTION

The notions of convex and concave functions are fundamental in mathematics. Formal definitions of these concepts are presented below. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ with $b > a$. A function $f : [a, b] \to \mathbb{R}$ is said to be convex if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \tag{1.1}$$

Conversely, a function $f : [a, b] \to \mathbb{R}$ is said to be concave if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y). \tag{1.2}$$

Convex functions are particularly essential in optimization theory, economics and functional analysis. One of their most significant consequences is the emergence of a diverse range of integral inequalities, collectively known as convex

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integral inequalities. The most well-known example is the Hermite-Hadamard integral inequality, which is presented below. Let $a, b \in \mathbb{R}$ such that $b > a$. Then, assuming that $f : [a, b] \to \mathbb{R}$ is a convex function, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.3}$$

If the function is concave rather than convex, then the double inequality is reversed, i.e.,

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f\left(\frac{a+b}{2}\right). \tag{1.4}$$

Convex integral inequalities continue to play a central role in several branches of mathematics, including numerical analysis, probability theory and information theory. Over the past few decades, many researchers have contributed to this field, establishing deeper connections between convexity, functional inequalities and integral transforms. Comprehensive discussions and recent developments on this topic can be found in [1–17].

This article contributes to the study of convex integral inequalities by presenting two new results of this kind. The proposed inequalities have a distinctive feature in that they depend on an adjustable parameter p and involve the integral

$$\int_0^1 x^p f(x)dx,$$

where $f : [0, 1] \to \mathbb{R}$ denotes the primary function under consideration. This integral is associated with the moments of f and may arise in various areas of analysis, probability theory and mathematical physics, particularly in problems involving weighted means, moment inequalities and functional approximation. To illustrate the theory, we will present several numerical examples, considering specific functions f and values of p.

The remainder of the article is composed of three sections: Section 2 and 3 describe the two new convex integral inequalities. A conclusion is given in Section 4.

2. First one-parameter convex integral inequality

The theorem below presents the first new convex integral inequality.

Theorem 2.1. *Let $p \in \mathbb{R}$ and $f : [0, 1] \to \mathbb{R}$ be a convex function.*

- *For $p \geq 0$, we have*

$$\int_0^1 x^p \left(f(x) + pf\left(\frac{x}{2}\right)\right) dx \leq \int_0^1 f(x)dx.$$

- *For $p < 0$, if $\lim_{x\to 0} x^p \int_0^x f(t)dt = 0$, then we have*

$$\int_0^1 x^p \left(f(x) + pf\left(\frac{x}{2}\right)\right) dx \geq \int_0^1 f(x)dx,$$

provided that the integrals exist.

If the function f is concave rather than convex, then the inequalities are reversed.

Proof. Let $p \geq 0$. For any $x \in [0, 1]$, let us introduce

$$F(x) = \int_0^x f(t)dt.$$

Applying one hand side of the Hermite-Hadamard integral inequality to the convex function f and the interval $[0, x]$ (see Equation (1.3)), for any $x \in [0, 1]$, we get

$$\frac{1}{x-0}\int_0^x f(t)dt \geq f\left(\frac{x+0}{2}\right),$$

which gives

$$F(x) \geq xf\left(\frac{x}{2}\right). \tag{2.1}$$

Making an integration by parts, and using Equation (2.1) and $p \geq 0$, we obtain

$$\begin{aligned}\int_0^1 x^p f(x)dx &= [x^p F(x)]_{x\to 0}^{x=1} - p\int_0^1 x^{p-1}F(x)dx \\ &= F(1) - p\int_0^1 x^{p-1}F(x)dx \\ &\leq F(1) - p\int_0^1 x^{p-1}xf\left(\frac{x}{2}\right)dx \\ &= \int_0^1 f(x)dx - p\int_0^1 x^p f\left(\frac{x}{2}\right)dx. \qquad (2.2)\end{aligned}$$

Rearranging this inequality, we get

$$\int_0^1 x^p \left(f(x) + pf\left(\frac{x}{2}\right)\right) dx \leq \int_0^1 f(x)dx.$$

If $p < 0$ rather than $p \geq 0$, taking into account the condition $\lim_{x\to 0} x^p F(x) = 0$, then the inequality in Equation (2.2) is reversed, and the final inequality is also reversed.

If the function f is concave rather than convex, then the inequality in Equation (2.1) is reversed and the final inequalities associated with $p \geq 0$ and $p < 0$ are also reversed. This completes the proof. □

Some examples of Theorem 2.1 in the convex case are given below.

- Taking $p = \pi$ and $f(x) = x^2$, $x \in [0, 1]$, which is convex, we get

$$\int_0^1 x^p \left(f(x) + pf\left(\frac{x}{2}\right)\right) dx = \int_0^1 x^\pi \left(x^2 + \pi\frac{x^2}{4}\right) dx \approx 0.290$$

  and

$$\int_0^1 f(x)dx = \int_0^1 x^2 dx \approx 0.333.$$

  It is evident that $0.290 < 0.333$, which illustrates the demonstrated inequality.
- Taking $p = 2$ and $f(x) = e^{-x}$, $x \in [0, 1]$, which is convex, we get

$$\int_0^1 x^p \left(f(x) + pf\left(\frac{x}{2}\right)\right) dx = \int_0^1 x^2 \left(e^{-x} + 2e^{-x/2}\right) dx \approx 0.621$$

  and

$$\int_0^1 f(x)dx = \int_0^1 e^{-x} dx \approx 0.632.$$

  It is clear that $0.621 < 0.632$, which demonstrates the inequality.
- Taking $p = 3$ and $f(x) = \sqrt{1 + x^2}$, $x \in [0, 1]$, which is convex, we get

$$\int_0^1 x^p \left(f(x) + pf\left(\frac{x}{2}\right)\right) dx = \int_0^1 x^3 \left(\sqrt{1 + x^2} + 3\sqrt{1 + \frac{x^2}{4}}\right) dx \approx 1.131$$

  and

$$\int_0^1 f(x)dx = \int_0^1 \sqrt{1 + x^2} dx \approx 1.147.$$

It is evident that $1.131 < 1.147$, which illustrates the demonstrated inequality.

- Let us now consider an example where $p < 0$. Taking $p = -1/2$ and $f(x) = e^x$, $x \in [0, 1]$, which is convex, we have

$$\lim_{x \to 0} x^p \int_0^x f(t)dt = \lim_{x \to 0} \frac{1}{\sqrt{x}} \int_0^x e^t dt = \lim_{x \to 0} \frac{1}{\sqrt{x}}(e^x - 1)$$
$$= \lim_{x \to 0} \sqrt{x} = 0.$$

The required condition is thus satisfied. Moreover, we have

$$\int_0^1 x^p \left(f(x) + pf\left(\frac{x}{2}\right)\right) dx = \int_0^1 \frac{1}{\sqrt{x}} \left(e^x - \frac{1}{2}e^{x/2}\right) dx \approx 1.730$$

and

$$\int_0^1 f(x)dx = \int_0^1 e^x dx \approx 1.718.$$

It is clear that $1.718 < 1.730$, which demonstrates the inequality.

Similar examples can be presented to support the theory of Theorem 2.1.

3. Second one-parameter convex integral inequality

The theorem below presents the second new convex integral inequality.

Theorem 3.1. *Let $p > -1$ and $f : [0, 1] \to \mathbb{R}$ be a concave function.*

- *For $p \geq 0$, we have*

$$\int_0^1 x^p f(x)dx \leq \frac{2}{p+2} \int_0^1 f(x)dx - \frac{p}{(p+2)(p+1)} f(0).$$

- *For $p \in (-1, 0)$, if $\lim_{x \to 0} x^p \int_0^x f(t)dt = 0$, then we have*

$$\int_0^1 x^p f(x)dx \geq \frac{2}{p+2} \int_0^1 f(x)dx - \frac{p}{(p+2)(p+1)} f(0),$$

provided that the integrals exist.

If the function f is convex rather than concave, then the inequalities are reversed.

Proof. Let $p \geq 0$. For any $x \in [0, 1]$, let us introduce

$$F(x) = \int_0^x f(t)dt.$$

Applying one hand side of the Hermite-Hadamard integral inequality to the concave function f and the interval $[0, x]$ (see Equation (1.4)), for any $x \in [0, 1]$, we get

$$\frac{1}{x-0}\int_0^x f(t)dt \geq \frac{1}{2}\left(f(x)+f(0)\right),$$

which gives

$$F(x) \geq \frac{x}{2}\left(f(x)+f(0)\right). \tag{3.1}$$

Making an integration by parts, and using Equation (3.1) and $p \geq 0$, we obtain

$$\begin{aligned}
\int_0^1 x^p f(x)dx &= [x^p F(x)]_{x\to 0}^{x=1} - p\int_0^1 x^{p-1}F(x)dx \\
&= F(1) - p\int_0^1 x^{p-1}F(x)dx \\
&\leq F(1) - p\int_0^1 x^{p-1}\frac{x}{2}\left(f(x)+f(0)\right)dx \\
&= \int_0^1 f(x)dx - \frac{p}{2}\int_0^1 x^p f(x)dx - \frac{p}{2}f(0)\int_0^1 x^p dx \\
&= \int_0^1 f(x)dx - \frac{p}{2}\int_0^1 x^p f(x)dx - \frac{p}{2}f(0)\left[\frac{x^{p+1}}{p+1}\right]_{x=0}^{x=1} \\
&= \int_0^1 f(x)dx - \frac{p}{2}\int_0^1 x^p f(x)dx - \frac{p}{2(p+1)}f(0). \qquad (3.2)
\end{aligned}$$

Therefore, we derive

$$\left(1+\frac{p}{2}\right)\int_0^1 x^p f(x)dx \leq \int_0^1 f(x)dx - \frac{p}{2(p+1)}f(0),$$

so that

$$\int_0^1 x^p f(x)dx \leq \frac{2}{p+2}\int_0^1 f(x)dx - \frac{p}{(p+2)(p+1)}f(0).$$

If $p \in (-1, 0)$ rather than $p \geq 0$, taking into account the condition $\lim_{x\to 0} x^p F(x) = 0$, then the inequality in Equation (3.2) is reversed, and the final inequality is also reversed.

If the function f is convex rather than concave, then the inequality in Equation (3.1) is reversed and the final inequalities associated with $p \geq 0$ and $p \in (-1, 0)$ are also reversed. This completes the proof. $\square$

In particular, if $f(0) \geq 0$ and $p \geq 0$, Theorem 3.1 in the concave case yields the elegant inequality

$$\int_0^1 x^p f(x)dx \leq \frac{2}{p+2} \int_0^1 f(x)dx.$$

If $p \in (-1, 0)$ rather than $p \geq 0$, then the inequality is reversed. To the best of our knowledge, this is a novel finding in the field of convex integral inequalities.

Some examples of Theorem 3.1 in the concave case are given below.

- Taking $p = \pi$ and $f(x) = \sqrt{x}$, $x \in [0, 1]$, which is concave, we get
$$\int_0^1 x^p f(x)dx = \int_0^1 x^\pi \sqrt{x}dx \approx 0.215$$
and
$$\frac{2}{p+2} \int_0^1 f(x)dx - \frac{p}{(p+2)(p+1)} f(0) = \frac{2}{\pi+2} \int_0^1 \sqrt{x}dx \approx 0.259.$$
It is clear that $0.215 < 0.259$, which demonstrates the inequality.
- Taking $p = 2$ and $f(x) = \ln(1 + x)$, $x \in [0, 1]$, which is concave, we get
$$\int_0^1 x^p f(x)dx = \int_0^1 x^2 \ln(1 + x)dx \approx 0.184$$
and
$$\frac{2}{p+2} \int_0^1 f(x)dx - \frac{p}{(p+2)(p+1)} f(0) = \frac{1}{2} \int_0^1 \ln(1 + x)dx \approx 0.193.$$
It is evident that $0.184 < 0.193$, which illustrates the demonstrated inequality.
- Taking $p = 3$ and $f(x) = \arctan(x)$, $x \in [0, 1]$, which is concave, we get
$$\int_0^1 x^p f(x)dx = \int_0^1 x^3 \arctan(x)dx \approx 0.166$$
and
$$\frac{2}{p+2} \int_0^1 f(x)dx - \frac{p}{(p+2)(p+1)} f(0) = \frac{2}{5} \int_0^1 \arctan(x)dx \approx 0.175.$$

It is clear that $0.166 < 0.175$, which demonstrates the inequality.

- Taking $p = 4$ and $f(x) = \sin(x)$, $x \in [0, 1]$, which is concave, we get

$$\int_0^1 x^p f(x)dx = \int_0^1 x^4 \sin(x)dx \approx 0.146$$

and

$$\frac{2}{p+2}\int_0^1 f(x)dx - \frac{p}{(p+2)(p+1)}f(0) = \frac{2}{6}\int_0^1 \sin(x)dx \approx 0.153.$$

It is clear that $0.146 < 0.153$, which demonstrates the inequality.

- Let us now consider an example where $p \in (-1, 0)$. Taking $p = -1/2$ and $f(x) = \cos(x)$, $x \in [0, 1]$, which is concave, we have

$$\lim_{x\to 0} x^p \int_0^x f(t)dt = \lim_{x\to 0} \frac{1}{\sqrt{x}} \int_0^x \cos(t)dt = \lim_{x\to 0} \frac{1}{\sqrt{x}} \sin(x)$$
$$= \lim_{x\to 0} \sqrt{x} = 0.$$

The required condition is thus satisfied. Moreover, we have

$$\int_0^1 x^p f(x)dx = \int_0^1 \frac{1}{\sqrt{x}} \cos(x)dx \approx 1.809$$

and

$$\frac{2}{p+2}\int_0^1 f(x)dx - \frac{p}{(p+2)(p+1)}f(0)$$
$$= \frac{2}{-1/2+2}\int_0^1 \cos(x)dx - \frac{(-1/2)}{(-1/2+2)(-1/2+1)} \approx 1.788.$$

It is clear that $1.788 < 1.809$, which demonstrates the inequality.

Similar examples can be presented to support the theory of Theorem 3.1.

4. Conclusion

In this article, we introduced two new one-parameter convex integral inequalities and demonstrated their usefulness using examples. The results highlight the flexibility offered by the adjustable parameter p. Future research could explore multidimensional extensions, inequalities involving different types of moments, or applications in optimization and probability theory. Investigating connections with fractional calculus and nonlinear functional inequalities also presents a promising direction for future research.

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