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ON COMPLEMENTARY ONE-PARAMETER CONVEX INTEGRAL INEQUALITIES

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ABSTRACT. In this article, we present additional results that build upon recently established convex integral inequalities. We provide several numerical examples to illustrate and validate the effectiveness of the findings.

1. INTRODUCTION

The concepts of convex and concave functions are fundamental to mathematics and its applications. For the sake of completeness, we present their formal definitions below. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ with $b > a$.

- A function $f : [a, b] \to \mathbb{R}$ is said to be convex if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \tag{1.1}$$

- A function $f : [a, b] \to \mathbb{R}$ is said to be concave if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y). \tag{1.2}$$

One of the most significant implications of convex functions is the establishment of a wide range of integral inequalities, often referred to as convex integral

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inequalities. The most prominent example of these is the Hermite-Hadamard integral inequality, which is stated below. Let $a, b \in \mathbb{R}$ such that $b > a$.

- Assuming that $f : [a, b] \to \mathbb{R}$ is a convex function, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.3}$$

- Assuming that $f : [a, b] \to \mathbb{R}$ is a concave function, we have

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f\left(\frac{a+b}{2}\right). \tag{1.4}$$

The double inequality is thus reversed in the concave case.

Convex integral inequalities remain at the heart of many areas of mathematics. Over the past few decades, substantial progress has been made, with numerous researchers uncovering deeper connections between convexity, functional inequalities and integral transforms. Comprehensive treatments and recent advances in this area can be found in [1–17].

In particular, in [6], the two theorems below are established.

Theorem 1.1. *[6, Theorem 2.1] Let $p \in \mathbb{R}$ and $f : [0, 1] \to \mathbb{R}$ be a convex function.*

- *For $p \geq 0$, we have*

$$\int_0^1 x^p \left(f(x) + pf\left(\frac{x}{2}\right)\right) dx \leq \int_0^1 f(x)dx.$$

- *For $p < 0$, if $\lim_{x\to 0} x^p \int_0^x f(t)dt = 0$, then we have*

$$\int_0^1 x^p \left(f(x) + pf\left(\frac{x}{2}\right)\right) dx \geq \int_0^1 f(x)dx,$$

provided that the integrals exist.

If the function f is concave rather than convex, then the inequalities are reversed.

Theorem 1.2. *[6, Theorem 3.1] Let $p > -1$ and $f : [0, 1] \to \mathbb{R}$ be a concave function.*

- *For $p \geq 0$, we have*

$$\int_0^1 x^p f(x)dx \leq \frac{2}{p+2}\int_0^1 f(x)dx - \frac{p}{(p+2)(p+1)}f(0).$$

- *For* $p \in (-1, 0)$*, if* $\lim_{x\to 0} x^p \int_0^x f(t)dt = 0$*, then we have*

$$\int_0^1 x^p f(x)dx \geq \frac{2}{p+2}\int_0^1 f(x)dx - \frac{p}{(p+2)(p+1)}f(0),$$

provided that the integrals exist.

If the function f *is convex rather than concave, then the inequalities are reversed.*

The aim of this article is to present integral inequalities that mirror these theorems. Three such results are demonstrated in detail. The theory is supported by numerous examples of convex and concave functions.

The remainder of the article is composed of three sections: Section 2, 3 and 4 describe the three new convex integral inequalities. Section 5 contains the conclusion.

2. First one-parameter convex integral inequality

The result below is inspired from [6, Theorem 2.1], considering the weight function $(1-x)^p$ instead of x^p.

Theorem 2.1. *Let* $p \in \mathbb{R}$ *and* $f : [0, 1] \to \mathbb{R}$ *be a convex function.*

- *For* $p \geq 0$*, we have*

$$\int_0^1 (1-x)^p \left(f(x) + pf\left(\frac{1+x}{2}\right)\right) dx \leq \int_0^1 f(x)dx.$$

- *For* $p < 0$*, if* $\lim_{x\to 1}(1-x)^p \int_x^1 f(t)dt = 0$*, then we have*

$$\int_0^1 (1-x)^p \left(f(x) + pf\left(\frac{1+x}{2}\right)\right) dx \geq \int_0^1 f(x)dx,$$

provided that the integrals exist.

If the function f *is concave rather than convex, then the inequalities are reversed.*

Proof. Let $p \geq 0$. For any $x \in [0, 1]$, let us define

$$G(x) = \int_x^1 f(t)dt.$$

Applying one hand side of the Hermite-Hadamard integral inequality to the convex function f and the interval $[x, 1]$ (see Equation (1.3)), for any $x \in [0, 1)$, we

get

$$\frac{1}{1-x}\int_x^1 f(t)dt \geq f\left(\frac{1+x}{2}\right),$$

which gives

$$G(x) \geq (1-x)f\left(\frac{1+x}{2}\right). \tag{2.1}$$

Doing an integration by parts, and using Equation (2.1), $G(1) = 0$ and $p \geq 0$, we obtain

$$\int_0^1 (1-x)^p f(x)dx = [(1-x)^p(-G(x))]_{x=0}^{x\to 1} \tag{2.2}$$

$$- p\int_0^1 (-(1-x)^{p-1})(-G(x))dx$$

$$= G(0) - p\int_0^1 (1-x)^{p-1}G(x)dx \tag{2.3}$$

$$\leq G(0) - p\int_0^1 (1-x)^{p-1}(1-x)f\left(\frac{1+x}{2}\right)dx$$

$$= \int_0^1 f(x)dx - p\int_0^1 (1-x)^p f\left(\frac{1+x}{2}\right)dx.$$

Rearranging this inequality, we get

$$\int_0^1 (1-x)^p \left(f(x) + pf\left(\frac{1+x}{2}\right)\right)dx \leq \int_0^1 f(x)dx.$$

If $p < 0$ rather than $p \geq 0$, taking into account the condition $\lim_{x\to 1}(1-x)^p G(x) = 0$, then the inequality in Equation (3.2) is reversed, and the final inequality is also reversed.

If the function f is concave rather than convex, then the inequality in Equation (2.1) is reversed and the final inequalities associated with $p \geq 0$ and $p < 0$ are also reversed. This completes the proof. □

The following are some examples of Theorem 2.1 in the convex case.

- Taking $p = \pi$ and $f(x) = x^2$, $x \in [0, 1]$, which is convex, we get

$$\int_0^1 (1-x)^p \left(f(x)+pf\left(\frac{1+x}{2}\right)\right)dx$$
$$=\int_0^1 (1-x)^\pi \left(x^2+\pi\left(\frac{1+x}{2}\right)^2\right)dx \approx 0.290$$

and

$$\int_0^1 f(x)dx = \int_0^1 x^2 dx \approx 0.333.$$

It is evident that $0.290 < 0.333$, which illustrates the demonstrated inequality.

- Taking $p=2$ and $f(x)=e^{-x}$, $x\in[0,1]$, which is convex, we get

$$\int_0^1 (1-x)^p \left(f(x)+pf\left(\frac{1+x}{2}\right)\right)dx$$
$$=\int_0^1 (1-x)^2 \left(e^{-x}+2e^{-(1+x)/2}\right)dx \approx 0.622$$

and

$$\int_0^1 f(x)dx = \int_0^1 e^{-x}dx \approx 0.632.$$

It is clear that $0.622 < 0.632$, which demonstrates the inequality.

- Taking $p=3$ and $f(x)=\sqrt{1+x^2}$, $x\in[0,1]$, which is convex, we get

$$\int_0^1 (1-x)^p \left(f(x)+pf\left(\frac{1+x}{2}\right)\right)dx$$
$$=\int_0^1 (1-x)^3 \left(\sqrt{1+x^2}+3\sqrt{1+\left(\frac{1+x}{2}\right)^2}\right)dx \approx 1.134$$

and

$$\int_0^1 f(x)dx = \int_0^1 \sqrt{1+x^2}dx \approx 1.147.$$

It is evident that $1.134 < 1.147$, which illustrates the demonstrated inequality.

- Let us now consider an example where $p < 0$. Taking $p = -1/2$ and $f(x) = e^x$, $x \in [0,1]$, which is convex, we have

$$\lim_{x\to 1}(1-x)^p \int_x^1 f(t)dt = \lim_{x\to 1} \frac{1}{\sqrt{1-x}} \int_x^1 e^t dt$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{1-x}}(e - e^x) = e \lim_{x\to 0} \sqrt{1-x} = 0.$$

The required condition is thus satisfied. Moreover, we have

$$\int_0^1 (1-x)^p \left(f(x) + pf\left(\frac{1+x}{2}\right)\right) dx$$

$$= \int_0^1 \frac{1}{\sqrt{1-x}} \left(e^x - \frac{1}{2} e^{(1+x)/2}\right) dx \approx 1.734$$

and

$$\int_0^1 f(x)dx = \int_0^1 e^x dx \approx 1.718.$$

It is clear that $1.718 < 1.734$, which demonstrates the inequality.

Similar examples can be presented to support the theory of Theorem 2.1.

3. Second one-parameter convex integral inequality

The result below complements Theorem 2.1 with a lower bound for $\int_0^1 x^p f(x)dx$.

Theorem 3.1. *Let $p \geq 0$ and $f : [0,1] \to \mathbb{R}$ be a convex function. Then we have*

$$\int_0^1 x^p f(x)dx \geq p \int_0^1 x^{p-1}(1-x) f\left(\frac{1+x}{2}\right) dx,$$

provided that the integrals exist.

If the function f is concave rather than convex, then the inequalities are reversed.

Proof. For any $x \in [0,1]$, let us define

$$G(x) = \int_x^1 f(t)dt.$$

Applying one hand side of the Hermite-Hadamard integral inequality to the convex function f and the interval $[x,1]$ (see Equation (1.3)), for any $x \in [0,1)$, we

get

$$\frac{1}{1-x}\int_x^1 f(t)dt \geq f\left(\frac{1+x}{2}\right),$$

which gives

$$G(x) \geq (1-x)f\left(\frac{1+x}{2}\right). \tag{3.1}$$

Doing an integration by parts, and using Equation (3.1), $G(1) = 0$ and $p \geq 0$, we obtain

$$\int_0^1 x^p f(x)dx = [x^p(-G(x))]_{x=0}^{x=1} - p\int_0^1 x^{p-1}(-G(x))dx$$

$$= p\int_0^1 x^{p-1}G(x)dx \geq p\int_0^1 x^{p-1}(1-x)f\left(\frac{1+x}{2}\right)dx. \tag{3.2}$$

If the function f is concave rather than convex, then the inequality in Equation (3.1) is reversed and the final inequality is also reversed. This completes the proof. □

Some examples of Theorem 3.1 in the convex case are given below.

- Taking $p = \pi$ and $f(x) = x^2$, $x \in [0, 1]$, which is convex, we get

  $$\int_0^1 x^p f(x)dx = \int_0^1 x^\pi x^2 dx \approx 0.162$$

  and

  $$p\int_0^1 x^{p-1}(1-x)f\left(\frac{1+x}{2}\right)dx$$

  $$= \pi\int_0^1 x^{\pi-1}(1-x)\left(\frac{1+x}{2}\right)^2 dx \approx 0.159.$$

  It is evident that $0.159 < 0.162$, which illustrates the demonstrated inequality.
- Taking $p = 2$ and $f(x) = e^{-x}$, $x \in [0, 1]$, which is convex, we get

  $$\int_0^1 x^p f(x)dx = \int_0^1 x^2 e^{-x} dx \approx 0.160$$

and

$$p\int_0^1 x^{p-1}(1-x)f\left(\frac{1+x}{2}\right)dx$$
$$= 2\int_0^1 x(1-x)e^{-(1+x)/2}dx \approx 0.158.$$

It is clear that $0.158 < 0.160$, which demonstrates the inequality.

- Taking $p = 3$ and $f(x) = \sqrt{1+x^2}$, $x \in [0,1]$, which is convex, we get

$$\int_0^1 x^p f(x)dx = \int_0^1 x^3\sqrt{1+x^2}dx \approx 0.321$$

and

$$p\int_0^1 x^{p-1}(1-x)f\left(\frac{1+x}{2}\right)dx$$
$$= 3\int_0^1 x^2(1-x)\sqrt{1+\left(\frac{1+x}{2}\right)^2}dx \approx 0.320.$$

It is evident that $0.320 < 0.321$, which illustrates the demonstrated inequality.

Similar examples can be presented to support the theory of Theorem 3.1.

4. Third one-parameter convex integral inequality

The result below is inspired from [6, Theorem 3.1], considering a new lower bound for $\int_0^1 x^p f(x)dx$.

Theorem 4.1. *Let $p \geq 0$ and $f : [0,1] \to \mathbb{R}$ be a concave function. Then we have*

$$\int_0^1 x^p f(x)dx \geq \frac{p}{p+2}\int_0^1 x^{p-1}f(x)dx + \frac{1}{(p+1)(p+2)}f(1).$$

If the function f is convex rather than concave, then the inequality is reversed.

Proof. For any $x \in [0,1]$, let us define

$$G(x) = \int_x^1 f(t)dt.$$

Applying one hand side of the Hermite-Hadamard integral inequality to the concave function f and the interval $[x,1]$ (see Equation (1.4)), for any $x \in [0,1)$, we

get

$$\frac{1}{1-x}\int_x^1 f(t)dt \geq \frac{1}{2}\left(f(x)+f(1)\right),$$

which gives

$$G(x) \geq \frac{1}{2}(1-x)\left(f(x)+f(1)\right). \tag{4.1}$$

Doing an integration by parts, and using Equation (4.1), $G(1) = 0$ and $p \geq 0$, we obtain

$$\begin{aligned}
&\int_0^1 x^p f(x)dx = [x^p(-G(x))]_{x=0}^{x=1} - p\int_0^1 x^{p-1}(-G(x))dx \\
&= p\int_0^1 x^{p-1}G(x)dx \\
&\geq p\int_0^1 x^{p-1}(1-x)\frac{1}{2}\left(f(x)+f(1)\right)dx \\
&= \frac{p}{2}\int_0^1 x^{p-1}f(x)dx - \frac{p}{2}\int_0^1 x^p f(x)dx + \frac{p}{2}f(1)\int_0^1 x^{p-1}(1-x)dx \\
&= \frac{p}{2}\int_0^1 x^{p-1}f(x)dx - \frac{p}{2}\int_0^1 x^p f(x)dx + \frac{p}{2}f(1)\left[\frac{x^p}{p} - \frac{x^{p+1}}{p+1}\right]_{x=0}^{x=1} \\
&= \frac{p}{2}\int_0^1 x^{p-1}f(x)dx - \frac{p}{2}\int_0^1 x^p f(x)dx + \frac{1}{2(p+1)}f(1).
\end{aligned}$$

Therefore, we derive

$$\left(1+\frac{p}{2}\right)\int_0^1 x^p f(x)dx \geq \frac{p}{2}\int_0^1 x^{p-1}f(x)dx + \frac{1}{2(p+1)}f(1),$$

so that

$$\int_0^1 x^p f(x)dx \geq \frac{p}{p+2}\int_0^1 x^{p-1}f(x)dx + \frac{1}{(p+1)(p+2)}f(1).$$

If the function f is convex rather than concave, then the inequality in Equation (4.1) is reversed and the final inequality is also reversed. This completes the proof. □

Some examples of Theorem 4.1 in the concave case are given below.

- Taking $p = \pi$ and $f(x) = \sqrt{x}$, $x \in [0, 1]$, which is concave, we get

$$\int_0^1 x^p f(x)dx = \int_0^1 x^\pi \sqrt{x}dx \approx 0.215$$

and

$$\frac{p}{p+2}\int_0^1 x^{p-1}f(x)dx + \frac{1}{(p+1)(p+2)}f(1)$$
$$= \frac{\pi}{\pi+2}\int_0^1 x^{\pi-1}\sqrt{x}dx + \frac{1}{(\pi+1)(\pi+2)} \approx 0.214.$$

It is clear that $0.214 < 0.215$, which demonstrates the inequality.

- Taking $p = 2$ and $f(x) = \ln(1 + x)$, $x \in [0, 1]$, which is concave, we get

$$\int_0^1 x^p f(x)dx = \int_0^1 x^2 \ln(1+x)dx \approx 0.184$$

and

$$\frac{p}{p+2}\int_0^1 x^{p-1}f(x)dx + \frac{1}{(p+1)(p+2)}f(1)$$
$$= \frac{1}{2}\int_0^1 x\ln(1+x)dx + \frac{1}{12}\ln(2) \approx 0.182.$$

It is evident that $0.182 < 0.184$, which illustrates the demonstrated inequality.

- Taking $p = 3$ and $f(x) = \arctan(x)$, $x \in [0, 1]$, which is concave, we get

$$\int_0^1 x^p f(x)dx = \int_0^1 x^3 \arctan(x)dx \approx 0.166$$

and

$$\frac{p}{p+2}\int_0^1 x^{p-1}f(x)dx + \frac{1}{(p+1)(p+2)}f(1)$$
$$= \frac{3}{5}\int_0^1 x^2 \arctan(x)dx + \frac{1}{20} \times \frac{\pi}{4} \approx 0.165.$$

It is clear that $0.165 < 0.166$, which demonstrates the inequality.

- Taking $p = 4$ and $f(x) = \sin(x)$, $x \in [0, 1]$, which is concave, we get

$$\int_0^1 x^p f(x)dx = \int_0^1 x^4 \sin(x)dx \approx 0.1466$$

and

$$\frac{p}{p+2}\int_0^1 x^{p-1}f(x)dx + \frac{1}{(p+1)(p+2)}f(1)$$
$$= \frac{2}{3}\int_0^1 x^3\sin(x)dx + \frac{1}{30}\sin(1) \approx 0.1461.$$

It is clear that $0.1461 < 0.1466$, which demonstrates the inequality. Similar examples can be presented to support the theory of Theorem 4.1.

5. Conclusion

In this article, we build upon the findings of [6] by presenting three new one-parameter convex integral inequalities. We demonstrate their sharpness and practical relevance through several numerical examples involving convex and concave functions.

Potential future research directions include investigating corresponding multi-parameter generalizations, developing discrete and fractional analogues, studying related inequalities for other generalized convexity classes, and exploring potential applications in numerical integration, optimization theory, and information measures.

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