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ON AN INTEGRAL REPRESENTATION OF THE EULER-MASCHERONI CONSTANT

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ABSTRACT. In this note, we investigate a classical integral representation of the Euler-Mascheroni constant. In particular, we establish a two-sided inequality that provides sharp bounds for the difference between a truncated integral and the constant itself.

1. INTRODUCTION

The Euler-Mascheroni constant is classically given by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right).$$

It admits several equivalent integral representations, one of which is given by

$$\gamma = -\int_0^{\infty} e^{-t}\ln t\,dt.$$

This identity arises naturally from the study of the gamma function and its logarithmic derivative at 1. See [1,2].

For any real number $a \geq 1$, let us define the truncated integral

$$I(a) = -\int_0^{a} e^{-t}\ln t\,dt.$$

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Clearly, $I(a) \to \gamma$ as $a \to \infty$, and $I(a)$ can be regarded as an approximation to γ obtained by restricting the upper limit of integration. It is then natural to ask how large the error term $I(a) - \gamma$ can be. This note provides an answer by establishing a precise two-sided inequality of this error term. A key to the proof is [1, p. 229, 5.1.20].

2. Main result

Our unique theorem is stated below. We emphasize the item (2), presenting our main two-sided inequality.

Theorem 2.1. *For any $a \geq 1$, we define*

$$I(a) = -\int_0^a e^{-t} \ln t dt.$$

Then the results below hold.

(1) *$I(a)$ is non-increasing on $[1, \infty)$, and $\lim_{a \to \infty} I(a) = \gamma$.*
(2) *For any $a \geq 1$ we have the two-sided inequality*

$$e^{-a} \ln \left(a \sqrt{1 + \frac{2}{a}} \right) \leq I(a) - \gamma \leq e^{-a} \ln(1 + a).$$

Proof.

(1) Differentiating the integral sign with respect to a gives

$$I'(a) = -\frac{d}{da} \int_0^a e^{-t} \ln t dt = -e^{-a} \ln a.$$

For $a \geq 1$, we have $\ln a \geq 0$, so $I'(a) \leq 0$. Hence $I(a)$ is non-increasing on $[1, \infty)$.

Let us set

$$J(a) = I(a) - \gamma.$$

By definition, we have

$$\gamma = -\int_0^\infty e^{-t} \ln t dt.$$

Splitting the integral at a gives

$$\gamma = -\int_0^a e^{-t}\ln t dt - \int_a^\infty e^{-t}\ln t dt = I(a) - \int_a^\infty e^{-t}\ln t dt.$$

Thus we have

$$J(a) = \int_a^\infty e^{-t}\ln t dt.$$

Because $e^{-t}\ln t \geq 0$ for $t \geq 1$, we have $J(a) \geq 0$, so $I(a) \geq \gamma$. As $a \to \infty$, we obtain $J(a) \to 0$, hence $I(a) \downarrow \gamma$.

(2) Let us investigate sharp bounds for $J(a)$.

Upper bound for $J(a)$: Integrating by parts with $u = \ln t$, $dv = e^{-t}dt$, so $du = dt/t$ and $v = -e^{-t}$, we obtain

$$J(a) = \big[-e^{-t}\ln t\big]_{t=a}^{t\to\infty} + \int_a^\infty \frac{e^{-t}}{t}dt = e^{-a}\ln a + \int_a^\infty \frac{e^{-t}}{t}dt. \tag{2.1}$$

Using [1, p. 229, 5.1.20], we have

$$\int_a^\infty \frac{e^{-t}}{t}dt \leq e^{-a}\ln\left(1+\frac{1}{a}\right).$$

It follows from this and Equation (2.1) that

$$I(a) - \gamma = J(a) \leq e^{-a}\left(\ln a + \ln\left(1+\frac{1}{a}\right)\right) = e^{-a}\ln(1+a).$$

This proves the stated upper bound.

Lower bound for $J(a)$: Let us investigate a lower bound for $J(a)$ based on the expression in Equation (2.1). Using [1, p. 229, 5.1.20], we have

$$\int_a^\infty \frac{e^{-t}}{t}dt \geq \frac{1}{2}e^{-a}\ln\left(1+\frac{2}{a}\right).$$

Substituting into the expression for $J(a)$ in Equation (2.1) yields

$$I(a) - \gamma = J(a) \geq e^{-a}\left(\ln a + \frac{1}{2}\ln\left(1+\frac{2}{a}\right)\right)$$
$$= e^{-a}\ln\left(a\sqrt{1+\frac{2}{a}}\right).$$

This proves the stated lower bound.

Combining the upper and lower bounds completes the proof.

This concludes the proof of the theorem. □

We emphasize the elegant inequality

$$I(a) - \gamma \leq e^{-a} \ln(1 + a).$$

It highlights that, for large a, the upper bound $e^{-a} \ln(1 + a)$ becomes extremely small because the exponential decay dominates the slow growth of the logarithm.

The two-sided inequality in Theorem 2.1 is illustrated graphically in Figure 1 and numerically in Table 1.

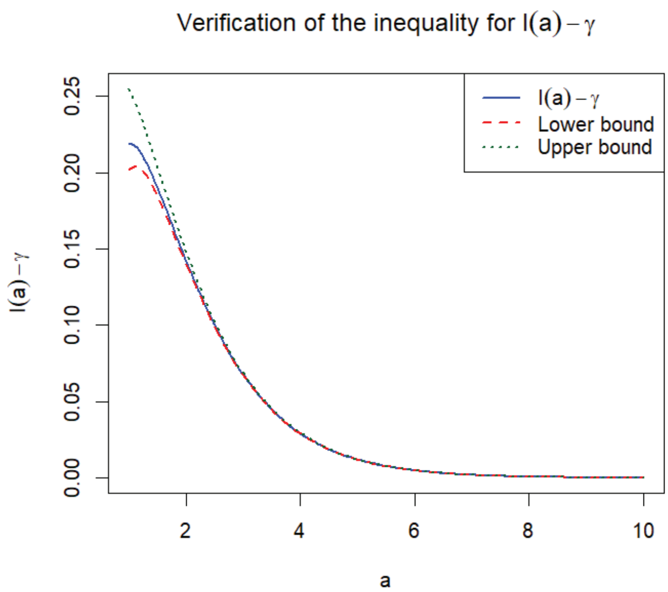

FIGURE 1. Illustration of the two-sided inequality in Theorem 2.1. The numerical evaluation of $I(a) - \gamma$ (blue curve) is shown together with the analytical lower and upper bounds (red and green dashed curves, respectively).

In this figure, we observe that the curves nearly coincide for $a \geq 3$, highlighting the sharpness of the derived inequality

TABLE 1. Numerical verification of the two-sided inequality in Theorem 2.1.

a	$I(a) - \gamma$	Lower bound	Upper bound
1	0.219384	0.202078	0.254995
2	0.142708	0.140711	0.148681
3	0.067745	0.067413	0.069020
5	0.011993	0.011978	0.012073
10	0.000109	0.000109	0.000109

This table clearly demonstrates the numerical sharpness of the derived inequality for selected values of a.

3. CONCLUSION

In conclusion, the derived two-sided inequality provides sharp bounds on the difference between a truncated integral and the Euler-Mascheroni constant. Numerical results confirm the accuracy of this inequality.

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