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SOME PARAMETRIC CLASSES OF COMPLETELY MONOTONIC FUNCTIONS INVOLVING POLYGAMMA FUNCTIONS

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ABSTRACT. This paper introduces some parametric classes of completely monotonic functions involving the psi and polygamma functions.

1. INTRODUCTION

The classical Euler gamma function is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0$$

and satisfies the functional relation

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0.$$

The function $\Psi(x) = \frac{d}{dx}\ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$, which is the logarithmic derivative of the gamma function, is called psi function or digamma function. Moreover, $\Psi^{(m)}(x)$ for $m \in \mathbb{N}$ are called a polygamma function. Properties and inequalities related to polygamma functions can be found in the recent publications [2, 4–6, 8] and the references therein.

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The integral representation of the function $\Psi(x)$ is given in [1, Page. 26], as follows:

(1.1) $$\Psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right) dt, \quad x > 0.$$

In addition, a non-negative function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and satisfies the inequality

$$(-1)^n f^{(n)}(x) \geq 0$$

for $x \in I$ and $n \in \mathbb{N} \cup \{0\}$. The famous Bernstein-Widder Theorem [7] states that a function f is completely monotonic on $(0, \infty)$ if and only if it can be represented in the form:

(1.2) $$f(x) = \int_0^\infty e^{-xt} d\varphi(t), \quad x > 0,$$

where $\varphi(t)$ is a non-decreasing function and the integral converges for all $x > 0$. The following corollary is a significant result in this area of study.

Corollary 1.1. *[3, Corollary 1] For $x > 0$, we have*

$$\frac{1}{2x} - \frac{1}{12x^2} < \Psi(x+1) - \ln(x) < \frac{1}{2x}$$

and

$$\frac{1}{2x^2} - \frac{1}{6x^3} < \frac{1}{x}\Psi'(x+1) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5}.$$

For $x > 0$ and for any non-negative integer m, we have

$$\frac{1}{x^{m+1}} = \frac{1}{m!} \int_0^\infty t^m e^{-xt} dt.$$

Using this, in [3, Equation 13], it is stated that

(1.3) $$\Psi(x) - \ln(x) + \frac{1}{x} = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-xt} dt.$$

Based on this, we introduce parametric classes of completely monotonic functions expressed in terms of psi and polygamma functions. These results make a valuable contribution to the broader study of special functions and their associated functional inequalities.

We are now in a position to present our main results.

2. Main Results

Theorem 2.1. *The function*

$$f_\alpha(x) = \ln(x) - \frac{\alpha}{x} - \Psi(x)$$

is completely monotonic on $(0, \infty)$ *if* $\alpha \leq \frac{1}{2}$.

Proof. Utilizing Equation (1.3), we obtain

$$\begin{aligned} -f_\alpha(x) = \Psi(x) - \ln(x) + \frac{\alpha}{x} &= \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-xt} dt + (\alpha - 1) \int_0^\infty e^{-xt} dt \\ &= \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} + \alpha - 1\right) e^{-xt} dt. \end{aligned}$$

We thus aim to apply the Bernstein-Widder Theorem, as described in Equation (1.2), thanks to this integral representation.

Let us set

$$\begin{aligned} h_\alpha(t) &= \frac{1}{t} - \frac{1}{e^t - 1} + \alpha - 1 \\ &= \frac{e^t - 1 - t + (\alpha - 1)t(e^t - 1)}{t(e^t - 1)} \end{aligned}$$

with the aim to show that $e^t - 1 - t + (\alpha - 1)t(e^t - 1) \leq 0$ for $\alpha \leq \frac{1}{2}$. Using the exponential series decomposition, we get

$$e^t - 1 - t + (\alpha - 1)t(e^t - 1) = \sum_{n=2}^{\infty} \left(\frac{1}{n!} + \frac{(\alpha - 1)}{(n-1)!}\right) t^n \leq 0$$

for $\alpha \leq \frac{1}{2}$. Hence, it follows from Equation (1.2) that the function $f_\alpha(x)$ is completely monotonic on $(0, \infty)$. □

Theorem 2.2. *The function*

$$f_{\alpha,\beta}(x) = \Psi(x) - \ln(x) + \frac{\alpha}{x} + \frac{\beta}{x^2}$$

is completely monotonic on $(0, \infty)$ *if* $\alpha \geq \frac{1}{2}$ *and* $\beta \geq \frac{1}{12}$.

Proof. Utilizing Equation (1.3), we obtain

$$\Psi(x) - \ln(x) + \frac{1}{x} = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-xt} dt,$$

so that

$$f_{\alpha,\beta}(x) = \Psi(x) - \ln(x) + \frac{\alpha}{x} + \frac{\beta}{x^2} = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-xt}dt + (\alpha - 1)\int_0^\infty e^{-xt}dt$$
$$+ \beta \int_0^\infty te^{-xt}dt$$
$$= \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1} + \alpha - 1 + \beta t\right) e^{-xt}dt.$$

We thus aim to apply the Bernstein-Widder Theorem, as described in Equation (1.2), thanks to this integral representation.

Let us set

$$h_{\alpha,\beta}(t) = \frac{1}{t} - \frac{1}{e^t - 1} + \alpha - 1 + \beta t$$
$$= \frac{e^t - 1 - t + (\alpha-1)t(e^t-1) + \beta t^2(e^t-1)}{t(e^t-1)}$$

with the aim to show that $e^t - 1 - t + (\alpha - 1)t(e^t - 1) + \beta t^2(e^t - 1) \geq 0$ for $\alpha \geq \frac{1}{2}$ and $\beta \geq \frac{1}{12}$. Using the exponential series decomposition, we get

$$e^t - 1 - t + (\alpha - 1)t(e^t - 1) + \beta t^2(e^t - 1)$$
$$= \left(\alpha - \frac{1}{2}\right) t^2 + \sum_{n=3}^{\infty} \left(\frac{1}{n!} + \frac{(\alpha-1)}{(n-1)!} + \frac{\beta}{(n-2)!}\right) t^n \geq 0$$

for $\alpha \geq \frac{1}{2}$, $\beta \geq \frac{1}{12}$ and $n \geq 3$. Indeed, for $\alpha \geq \frac{1}{2}$, we obtain

$$\frac{1}{n!} + \frac{(\alpha-1)}{(n-1)!} + \frac{\beta}{(n-2)!} \geq \frac{-1}{2(n-1)!} + \frac{1}{n!} + \frac{\beta}{(n-2)!}$$
$$= \frac{1}{2n!}\left[2\beta n^2 - (2\beta+1)n + 2\right] \geq 0$$

for all $n \geq 3$ and if $\beta \geq \frac{1}{12}$. By Equation (1.2), the function $f_{\alpha,\beta}(x)$ is completely monotonic on $(0, \infty)$. $\square$

Theorem 2.3. *The function*

$$f_\alpha(x) = \frac{\alpha}{x} - \frac{1}{x^3} - \Psi''(x)$$

is completely monotonic on $(0, \infty)$ *if* $\alpha \geq 0$.

Proof. Using the integral representation in Equation (1.1), we obtain

$$f_\alpha(x) = \frac{\alpha}{x} - \frac{1}{x^3} - \Psi''(x) = \alpha\int_0^\infty e^{-xt}dt - \frac{1}{2}\int_0^\infty t^2e^{-xt}dt + \int_0^\infty \frac{t^2}{1-e^{-t}}e^{-xt}dt$$
$$= \int_0^\infty \left(\alpha - \frac{t^2}{2} + \frac{t^2e^t}{e^t-1}\right)e^{-xt}dt.$$

We thus aim to apply the Bernstein-Widder Theorem, as described in Equation (1.2), thanks to this integral representation.

It remains to show that $\alpha - \frac{t^2}{2} + \frac{t^2e^t}{e^t-1} \geq 0$ for all $\alpha \geq 0$ and $t \geq 0$. Using the exponential series decomposition, we get

$$\alpha - \frac{t^2}{2} + \frac{t^2e^t}{e^t-1} = \frac{1}{2(e^t-1)}\left[2\alpha t + (\alpha+2)t^2 + \sum_{n=3}^{\infty}\left(\frac{2\alpha}{n!} + \frac{1}{(n-2)!}\right)t^n\right] \geq 0$$

Hence, it follows from Equation (1.2) that the function $f_\alpha(x)$ is completely monotonic on $(0,\infty)$. □

Theorem 2.4. *The function*

$$f_\alpha(x) = \frac{\alpha}{x^2} - \frac{1}{x^3} - \Psi''(x)$$

is completely monotonic on $(0,\infty)$ *if* $\alpha \geq 0$.

Proof. Using the integral representation in Equation (1.1), we obtain

$$f_\alpha(x) = \frac{\alpha}{x^2} - \frac{1}{x^3} - \Psi''(x) = \alpha\int_0^\infty te^{-xt}dt - \frac{1}{2}\int_0^\infty t^2e^{-xt}dt + \int_0^\infty \frac{t^2}{1-e^{-t}}e^{-xt}dt$$
$$= \int_0^\infty \left(\alpha t - \frac{t^2}{2} + \frac{t^2e^t}{e^t-1}\right)e^{-xt}dt.$$

We thus aim to apply the Bernstein-Widder Theorem, as described in Equation (1.2), thanks to this integral representation.

It remains to show that $\alpha t - \frac{t^2}{2} + \frac{t^2e^t}{e^t-1} \geq 0$ for all $\alpha \geq 0$ and $t \geq 0$. Using the exponential series decomposition, we get

$$(2\alpha t - t^2)(e^t-1) + 2t^2e^t$$
$$= \frac{1}{2(e^t-1)}\left[(2\alpha+2)t^2 + \sum_{n=3}^{\infty}\left(\frac{2\alpha}{(n-1)!} + \frac{1}{(n-1)!}\right)t^n\right] \geq 0.$$

Hence, it follows from Equation (1.2) that the function $f_\alpha(x)$ is completely monotonic on $(0,\infty)$. □

3. Conclusion

This paper establishes the complete monotonicity of several parametric classes of functions involving the psi and polygamma functions by employing their integral representations.

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References

[1] G.E. Andrews, R. Askey, R. Roy: *Special Functions*, Cambridge Univ. Press, Cambridge, 1999.

[2] C. Berg, A. Çetinkaya, D. Karp: *Completely monotonic ratios of basic and ordinary gamma functions*, Aequat. Math. (2021).

[3] F. Qi, R.-Q. Cui, C.-P. Chen, B.-N. Guo: *Some completely monotonic functions involving polygamma functions and an application*, J. Math. Anal. Appl., **310** (2005), 303–308.

[4] F. Qi, R.P. Agarwal: *On complete monotonicity for several classes of functions related to ratios of gamma functions*, J. Inequal. Appl. (2019), **2019:36**, 42 pp.

[5] F. Qi: *Necessary and sufficient condition for a ratio involving trigamma and tetragamma functions to be monotonic*, Turkish J. Inequal., **5**(1) (2021), 1–10.

[6] B. Ravi, A. Venkata Lakshmi: *Completely monotonicity of class functions involving the polygamma and related functions*, Asian-European J. Math., **14**(4) (2021), 2150064.

[7] D.V. Widder: *The Laplace Transform*, Princeton Mathematical Series, vol. **6**, Princeton University Press, Princeton, N.J., 1941.

[8] N.H. Zhou, D.W. Niu: *Monotonicity properties related to the ratio of two gamma functions*, Mediterr. J. Math., **18** (2021), 104.

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