

SOME COMPLETELY MONOTONIC FUNCTIONS ASSOCIATED WITH THE RATIO OF GAMMA FUNCTIONS

Bhukya Ravi and Christophe Chesneau¹

ABSTRACT. This paper introduces some monotonic functions associated with ratios of gamma functions.

1. INTRODUCTION

We begin by revisiting the key concepts of our study. A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I which alternate successively in sign, that is,

$$(1.1) \quad (-1)^n f^{(n)}(x) \geq 0$$

for all $x \in I$ and for all $n \geq 0$. If inequality in Equation (1.1) is strict for all $x \in I$ and all $n \geq 0$, then f is said to be strictly completely monotonic. The famous Bernstein-Widder Theorem [16] states that a function f is completely monotonic on $(0, \infty)$ if and only if it can be represented in the form:

$$(1.2) \quad f(x) = \int_0^\infty e^{-xt} d\alpha(t), \quad x > 0,$$

where $\alpha(t)$ is a non-decreasing function and the integral converges for all $x > 0$. The following lemma will be at the heart of our proofs.

¹corresponding author

Key words and phrases. Gamma function, Digamma function, completely monotonic function.

Submitted: 03.03.2026; *Accepted:* 19.03.2026; *Published:* 25.03.2026.

Lemma 1.1. [5] *The function $e^{-h(x)}$ is completely monotonic on an interval I if $h'(x)$ is completely monotonic on I .*

An extension of Lemma 1.1 can be found in [4, 8]. Completely monotonic functions play an important role in areas such as mathematical analysis [16], probability theory [3], numerical analysis [2], physics [11], and the theory of special functions.

The Euler gamma function defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \quad x > 0,$$

is one of the most important functions in analysis and its applications. There is an extensive literature on the gamma function. In particular, inequalities involving the gamma function and its logarithmic derivative $\psi = \Gamma'/\Gamma$ have been studied by many authors [1–3, 5, 11, 14]. The integral representation of ψ is given by [9], as follows:

$$(1.3) \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt, \quad x > 0.$$

In this paper, we introduce classes of completely monotonic functions expressed in terms of the psi and polygamma functions. These results contribute to the broader study of special functions and their associated functional inequalities.

We are now in a position to present our main results.

2. MAIN RESULTS

Theorem 2.1. *The function*

$$(2.1) \quad f(x) = \frac{\Gamma(x)^5 \Gamma(x+a+b+c)}{\Gamma(x+a)\Gamma(x+b)\Gamma(x+c)\Gamma(x+a+b)\Gamma(x+a+c)\Gamma(x+b+c)},$$

$$a, b, c \geq 0,$$

is completely monotonic on $(0, \infty)$.

Proof. Let us set $h(x) = -\log f(x)$. Then we have

$$h'(x) = \psi(x+a) + \psi(x+b) + \psi(x+c) + \psi(x+a+b) + \psi(x+a+c) + \psi(x+b+c) \\ - 5\psi(x) - \psi(x+a+b+c).$$

Using Equation (1.3), we obtain the integral representation

$$h'(x) = \int_0^\infty \frac{e^{-xt}}{1-e^{-t}} \left(5 + e^{-t(a+b+c)} - e^{-ta} - e^{-tb} - e^{-tc} \right. \\ \left. - e^{-t(a+b)} - e^{-t(a+c)} - e^{-t(b+c)} \right) dt \\ = \int_0^\infty \frac{e^{-xt}}{1-e^{-t}} \left((1-e^{-t(b+c)})(1-e^{-ta}) \right. \\ \left. + 4 - e^{-tb} - e^{-tc} - e^{-t(a+b)} - e^{-t(a+c)} \right) dt.$$

This shows that $h'(x)$ is completely monotonic on $(0, \infty)$. Hence, by Lemma 1.1 and Equation (1.2), the function $e^{-h(x)} = f(x)$ is also completely monotonic on $(0, \infty)$. \square

Theorem 2.2. For all $n_1, n_2, n_3 \in \mathbb{N}$ such that $n = n_1 + n_2 + n_3$, the function

$$(2.2) \quad f(x) = \frac{\Gamma(x)^n \Gamma(x+a+b+c)^{n_1}}{\Gamma(x+a)^{n_2} \Gamma(x+b)^{n_3} \Gamma(x+c)^{n_1} \Gamma(x+a+b)^{n_1}}, \quad a, b, c \geq 0,$$

is completely monotonic on $(0, \infty)$.

Proof. Let us set $h(x) = -\log f(x)$. Then we have

$$h'(x) = n_2\psi(x+a) + n_3\psi(x+b) + n_1\psi(x+c) + n_1\psi(x+a+b) \\ - n\psi(x) - n_1\psi(x+a+b+c).$$

Using Equation (1.3), we obtain the integral representation

$$h'(x) = \int_0^\infty \frac{e^{-xt}}{1-e^{-t}} \left(n + n_1 e^{-t(a+b+c)} - n_2 e^{-ta} - n_3 e^{-tb} \right. \\ \left. - n_1 e^{-tc} - n_1 e^{-t(a+b)} \right) dt \\ = \int_0^\infty \frac{e^{-xt}}{1-e^{-t}} \left(n_1 (1 - e^{-t(a+b)})(1 - e^{-tc}) \right. \\ \left. + n_2 (1 - e^{-ta}) + n_3 (1 - e^{-tb}) \right) dt.$$

This proves that $h'(x)$ is completely monotonic on $(0, \infty)$. Hence, by Lemma 1.1 and Equation (1.2), the function $e^{-h(x)} = f(x)$ is completely monotonic on $(0, \infty)$. \square

Theorem 2.3. For all $n_i \in \mathbb{N}$ with $n = \sum_{i=1}^k n_i$ and $a_i \geq 0$, the function

$$f(x) = \frac{\Gamma(x)^n \Gamma(x + \sum_{i=1}^k a_i)^{n_k}}{\prod_{i=1}^k \Gamma(x + a_i)^{n_i} \Gamma(x + \sum_{i=1}^{k-1} a_i)^{n_k}}$$

is completely monotonic on $(0, \infty)$.

Proof. Let us set $h(x) = -\log f(x)$. Then we have

$$h'(x) = \sum_{i=1}^k n_i \psi(x + a_i) + n_k \psi\left(x + \sum_{i=1}^{k-1} a_i\right) - n \psi(x) - n_k \psi\left(x + \sum_{i=1}^k a_i\right).$$

Using Equation (1.3), we obtain the integral representation

$$\begin{aligned} h'(x) &= \int_0^\infty \frac{e^{-xt}}{1 - e^{-t}} \left(n + n_k e^{-t(a_1 + a_2 + \dots + a_k)} \right. \\ &\quad \left. - \sum_{i=1}^k n_i e^{-ta_i} - n_k e^{-t(a_1 + a_2 + \dots + a_{k-1})} \right) dt \\ &= \int_0^\infty \frac{e^{-xt}}{1 - e^{-t}} \left(n_k (1 - e^{-ta_k}) (1 - e^{-t(a_1 + \dots + a_{k-1})}) \right. \\ &\quad \left. + \sum_{i=1}^{k-1} n_i (1 - e^{-ta_i}) \right) dt. \end{aligned}$$

This proves that $h'(x)$ is completely monotonic on $(0, \infty)$. Hence, by Lemma 1.1 and Equation (1.2), the function $e^{-h(x)} = f(x)$ is completely monotonic on $(0, \infty)$. \square

In 1997, H. Alzer [1] proved that the function

$$f(x) = \prod_{k=1}^n \frac{\Gamma(x + a_k)}{\Gamma(x + b_k)}$$

is completely monotonic on $(0, \infty)$. The following theorem provides an extension of this result.

Theorem 2.4. Let $a_k, b_k,$ and c_k be real numbers such that $c_k > 0$ and $0 \leq a_k \leq b_k$ for all $k = 1, \dots, n$. Then the function

$$(2.3) \quad f(x) = \prod_{k=1}^n \left(\frac{\Gamma(x + a_k)}{\Gamma(x + b_k)} \right)^{c_k}$$

is completely monotonic on $(0, \infty)$.

Proof. Let us set $h(x) = -\log f(x)$. Then we have

$$h'(x) = \sum_{k=1}^n c_k (\psi(x + a_k) - \psi(x + b_k)).$$

Using Equation (1.3), we obtain the integral representation

$$\begin{aligned} h'(x) &= \sum_{k=1}^n c_k \int_0^\infty \frac{e^{-xt}}{1 - e^{-t}} (e^{-tb_k} - e^{-ta_k}) dt \\ &= \int_0^\infty \frac{e^{-xt}}{1 - e^{-t}} \left(\sum_{k=1}^n c_k (e^{-tb_k} - e^{-ta_k}) \right) dt. \end{aligned}$$

This proves that $h'(x)$ is completely monotonic on $(0, \infty)$. Hence, by Lemma 1.1 and Equation (1.2), the function $e^{-h(x)} = f(x)$ is also completely monotonic on $(0, \infty)$. \square

3. CONCLUSION

This paper establishes several new classes of completely monotonic functions involving ratios of gamma functions. These results extend existing monotonicity properties and could prove useful when studying inequalities related to special functions.

ACKNOWLEDGMENT

The authors would like to thank the reviewers for their constructive comments.

REFERENCES

- [1] H. ALZER: *On some inequalities for gamma and psi functions*, Math. Comp., **66** (1997), 373–389.
- [2] H. ALZER: *Some gamma function inequalities*, Math. Comp., **60** (1993), 337–346.

- [3] G.D. ANDERSON, S.L. QIU: *A monotonicity property of the gamma function*, Proc. Amer. Math. Soc., **125** (1997), 3355–3362.
- [4] S. BOCHNER: *Harmonic analysis and the theory of probability*, Univ. of California Press, Berkeley–Los Angeles, 1955.
- [5] J. BUSTOZ, M.E.H. ISMAIL: *On gamma function inequalities*, Math. Comp., **46** (1986), 659–667.
- [6] W.A. DAY: *On monotonicity of the relaxation functions of viscoelastic materials*, Proc. Cambridge Philos. Soc., **67** (1970), 503–508.
- [7] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, F.G. TRICOMI: *Higher transcendental functions*, Vol. 1, McGraw-Hill, New York, 1953.
- [8] W. FELLER: *An introduction to probability theory and its applications*, Vol. 2, Wiley, New York, 1966.
- [9] L. GORDON: *A stochastic approach to the gamma function*, Amer. Math. Monthly, **101** (1994), 858–865.
- [10] A.Z. GRINSHPAN, M.E.H. ISMAIL: *Completely monotonic functions involving the gamma and the q -gamma functions*, Proc. Amer. Math. Soc., **134** (2005), 1153–1160.
- [11] M.E.H. ISMAIL: *Completely monotonic functions associated with the gamma function and its q -analogues*, J. Math. Anal. Appl., **116** (1986), 1–9.
- [12] L. MALIGRANDA, J.E. PEČARIĆ, L.E. PERSSON: *Stolarsky's inequality with general weights*, Proc. Amer. Math. Soc., **123** (1995), 2113–2118.
- [13] A.W. MARSHALL, I. OLKIN: *Inequalities: theory of majorization and its applications*, Academic Press, New York, 1979.
- [14] M.E. MULDOON: *Some monotonicity properties and characterizations of the gamma function*, Aequationes Math., **18** (1978), 54–63.
- [15] B. RAVI, A. VENKATA LAKSHMI: *Completely monotonicity of class functions involving the polygamma and related functions*, Asian-European J. Math., **14**(4) (2021), 2150064.
- [16] D.V. WIDDER: *The Laplace transform*, Princeton Univ. Press, Princeton, NJ, 1944.
- [17] J. WIMP: *Sequence transformations and their applications*, Academic Press, New York, 1981.

GOVERNMENT COLLEGE FOR MEN - ANANTAPUR, 515001, ANDHRA PRADESH, INDIA.

Email address: ravidevi19@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CAEN - NORMANDIE, UFR DES SCIENCES - CAMPUS 2, 14 000, CAEN, FRANCE.

Email address: christophe.chesneau@gmail.com