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ON A REFINEMENT ON AN INEQUALITY INVOLVING π AND e

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ABSTRACT. This study is based on a recent article and aims to refine an inequality involving π and e. Rigorous numerical work supports the findings.

1. INTRODUCTION

In a recent article, C. Chesneau [3] refined the famous inequality $\pi^e < e^\pi$ by using an integral approach. Moreover, using the same approach, he derived another elegant inequality, $e^{4\pi} < (\pi e)^{\pi+e}$.

His approach consists of introduce a variable $x > e$ and a function $f(x) \in (0,1)$ such that

$$f(x) = e^{\frac{-e(\log x-1)^2}{2}} \geq \frac{1}{g(x)} = \frac{x^e}{e^x}.$$

The use of f is to bridge the gap between π^e and e^π.

A function f on $(0,\infty)$ is said to be completely monotone if it possesses derivatives $f^{(n)}$ of all orders and, for each $x > 0$,

$$(-1)^n f^{(n)}(x) > 0.$$

Typical examples of a completely monotone function are e^x and $(l+x)^{-m}$, where m is a positive number.

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We have that if f and g are completely monotone then the product fg is also monotone and that if f is completely monotone and g is a positive function with completely monotone derivative then fog is completely monotone (in particular, e^{-g} is completely monotone).

Let

$$S_n(x) = \sum_{r=0}^{n} f^{(r)}(0)\frac{x^r}{r!}, \tag{1.1}$$

denote the partial sum of the Taylor series expansion of $f(x)$. The following theorem shows that if f is completely monotone then for each x the value of $f(x)$ lies between any two consecutive partial sums.

The following result yields approximation of completely monotone functions

Theorem 1.1. *[1] Let n be a positive integer. If f is completely monotone then, for each $x > 0$,*

$$S_{2n-1}(x) < f(x) < S_{2n-2}(x). \tag{1.2}$$

Proof. Let

$$h(x) = f(x) - S_{2n-1}(x).$$

We have $h^{(r)}(0) = 0$ for $r = 0, 1, 2, 3, \ldots, 2n-1$. Since $h^{(2n-1)}(0) = 0$ then $h^{(2n-1)}(x) > 0$. Repeating the argument we have that $h^{(2n-2)}(x) > 0$.
Successive repetition of the argument yields $h(x) > 0$, establishing the first inequality in (1.2). The second inequality in (1.2) is proved similarly. □

We easily prove the following

Theorem 1.2. *[2] For $x \in (e, \pi)$ the following functions*

$$g(x) = \frac{e^x}{x^e}, \qquad h(x) = \frac{e^{4x}}{(xe)^{x+e}}$$

are logarithmically monotonic function. In the sense for any $x \in (e, \pi)$ its logarithmic derivatives satisfy

$$(-1)^k[\log g(x)]^{(k)} \geq 0.$$

Proof. Indeed, the functions

$$\tilde{g}(x) = \log g(x) = x - e\log x, \qquad \tilde{h}(x) = 4x - (x+e)\log(xe)$$

are completely monotonic. □

Moreover, it is well known that any logarithmically monotonic function is also completely monotonic functions. But the converse is not true.

Compute now the successive derivatives of $g(x)$. We have

$$\begin{aligned}
g'(x) &= \frac{e^x\,(x-e)}{x^e x}, \quad g''(x) = \frac{e^x\left((x-e)^2+e\right)}{x^e x^2},\\
g^{(3)}(x) &= \frac{e^x\left((x-e)^3+3ex-3e^2-2e\right)}{x^e x^3},\\
g^{(4)}(x) &= \frac{e^x\left((x-e)^4+6ex^2-12e^2x-8ex+6e^3+11e^2+6e\right)}{x^e x^4},\dots
\end{aligned}$$

Let us define $\mathbf{S_n(x)}$ as in (1.1) with the change of variable x to $x-e$.

We then deduce

$$\mathbf{S_1(x)} = 0,$$

and, adopting an inline notation for the fractional, i.e., for instance, $1/2e^{-1} = 0.5\times e^{-1}$, we have

$$\begin{aligned}
\mathbf{S_2(x)} &= 1+1/2e^{-1}\,(x-e)^2\,,\ \mathbf{S_3(x)} = 1+1/2e^{-1}\,(x-e)^2-1/3e^{-2}\,(x-e)^3\,,\\
\mathbf{S_4(x)} &= 1+1/2e^{-1}\,(x-e)^2-1/3e^{-2}\,(x-e)^3+\left(1/8e^{-2}+1/4e^{-3}\right)(x-e)^4\,,\\
\mathbf{S_5(x)} &= 1+1/2e^{-1}\,(x-e)^2-1/3e^{-2}\,(x-e)^3\\
&\quad+\left(1/8e^{-2}+1/4e^{-3}\right)(x-e)^4+\left(-1/6e^{-3}-1/5e^{-4}\right)(x-e)^5\,,\\
\mathbf{S_6(x)} &= 1+1/2e^{-1}\,(x-e)^2-1/3e^{-2}\,(x-e)^3\\
&\quad+\left(1/8e^{-2}+1/4e^{-3}\right)(x-e)^4+\left(-1/6e^{-3}-1/5e^{-4}\right)(x-e)^5\\
&\quad+\left(1/48e^{-3}+13/72e^{-4}+1/6e^{-5}\right)(x-e)^6\,,\\
\mathbf{S_7(x)} &= 1+1/2e^{-1}\,(x-e)^2-1/3e^{-2}\,(x-e)^3\\
&\quad+\left(1/8e^{-2}+1/4e^{-3}\right)(x-e)^4+\left(-1/6e^{-3}-1/5e^{-4}\right)(x-e)^5\\
&\quad+\left(1/48e^{-3}+13/72e^{-4}+1/6e^{-5}\right)(x-e)^6\\
&\quad+\left(-11/60e^{-5}-1/7e^{-6}-1/24e^{-4}\right)(x-e)^7\,,\dots
\end{aligned}$$

Therefore, by Theorem 1.1, it follows that

$$\mathbf{S_1(x)} = 0 < g(x) < \mathbf{S_2(x)},$$

$$\mathbf{S_3(x)} < g(x) < \mathbf{S_4(x)}, \quad \mathbf{S_5(x)} < g(x) < \mathbf{S_6(x)}, \quad \mathbf{S_7(x)} < g(x) < \mathbf{S_8(x)}, \dots$$

We then deduce

$$\mathbf{S_2}(\pi) = 1 + 1/2e^{-1}\left(\pi - e\right)^2 \approx 1.032960537,$$

$$\mathbf{S_3}(\pi) = 1 + 1/2e^{-1}\left(\pi - e\right)^2 - 1/3e^{-2}\left(\pi - e\right)^3 \approx 1.029538632,$$

$$\begin{aligned}\mathbf{S_4}(\pi) &= 1 + 1/2e^{-1}\left(\pi - e\right)^2 - 1/3e^{-2}\left(\pi - e\right)^3 \\ &\quad + \left(1/8e^{-2} + 1/4e^{-3}\right)\left(\pi - e\right)^4 \approx 1.030481494,\end{aligned}$$

$$\begin{aligned}\mathbf{S_5}(\pi) &= 1 + 1/2e^{-1}\left(\pi - e\right)^2 - 1/3e^{-2}\left(\pi - e\right)^3 + \left(1/8e^{-2} + 1/4e^{-3}\right)\left(\pi - e\right)^4 \\ &\quad + \left(-1/6e^{-3} - 1/5e^{-4}\right)\left(\pi - e\right)^5 \approx 1.030318915,\end{aligned}$$

$$\begin{aligned}\mathbf{S_6}(\pi) &= 1 + 1/2e^{-1}\left(\pi - e\right)^2 - 1/3e^{-2}\left(\pi - e\right)^3 + \left(1/8e^{-2} + 1/4e^{-3}\right)\left(\pi - e\right)^4 \\ &\quad + \left(-1/6e^{-3} - 1/5e^{-4}\right)\left(\pi - e\right)^5 + \left(1/48e^{-3} + 13/72e^{-4} + 1/6e^{-5}\right)\left(\pi - e\right)^6 \\ &\approx 1.030350372, \ldots\end{aligned}$$

Say that

$$g(\pi) = \frac{e^{\pi}}{\pi^{e}} \approx 1.030345525.$$

We thus obtain a more accurate approximation than the one given by the function in [3], i.e.,

$$\tilde{f}(x) = e^{\frac{e(\log x - 1)^2}{2}} = \frac{1}{f(x)},$$

where $f(\pi) \approx 0.9719318684$ and $\tilde{f}(\pi) = 1.028878703 < \mathbf{S_3}(\pi)$.

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