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ON THREE NEW TRIGONOMETRIC UNIT DISTRIBUTIONS

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ABSTRACT. New and simple unit distributions are scarce in the recent literature. In this article, we introduce three such distributions with a trigonometric structure. For each distribution, we derive the cumulative distribution function, probability density function, and quantile function. The aim is to establish a theoretical foundation that can stimulate further developments, including applications in data analysis and extensions to regression frameworks.

1. INTRODUCTION

Unit distributions are a family of (probability) distributions with support the unit interval $(0, 1)$. They play a central role in modeling proportions, rates, and fractional data, where observations are inherently bounded between 0 and 1. Such data frequently arise in areas like economics (e.g., market shares), biology (e.g., gene expression levels), and reliability analysis (e.g., failure rates). A canonical example is the beta distribution, widely used for its analytical tractability and interpretability. Beyond the beta family, numerous extensions and alternative unit distributions have been developed to accommodate more complex features, such as boundary inflation, multimodality, or heavy tails. Furthermore, unit distributions are commonly incorporated into regression frameworks, known as unit regression models, where parameters, particularly the mean, are

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linked to covariates. This enhances their usefulness for analyzing bounded response variables in applied settings.

Recently, most existing unit distributions have been catalogued in a comprehensive survey in [1]; we refer the reader to this work and the many references therein. It appears that new and simple unit distributions remain scarce in the recent literature, particularly those with a trigonometric structure. We may mention the 2026 study in [2]. The aim of this article is to introduce new trigonometric unit distributions. Three distinct construction approaches are proposed: the first based on the cosine function, the second on the tangent function, and the third on the sine function. For each approach, we derive the fundamental distributional properties, with particular emphasis on the cumulative distribution function (CDF), probability density function (PDF), and quantile function (QF). The objective is to establish a tractable theoretical foundation that may stimulate further developments, including applications to data analysis and extensions to regression frameworks.

The rest of the article is articulated as follows: The construction approaches are developed in Section 2. A conclusion is given in Section 3.

2. Contruction approaches

2.1. **First approach.** The first approach is based on the cosine function and relies on the CDF introduced in the theorem below.

Theorem 2.1. *The following function is a valid CDF of a unit distribution:*

$$F_1(x) = 2\cos\left(\frac{\pi}{2+x}\right), \quad x \in (0,1),$$

which is completed with $F_1(x) = 0$ *for* $x \leq 0$ *and* $F_1(x) = 1$ *for* $x \geq 1$.

Proof. Let us analyze the continuity of F_1 at $x = 0$ and $x = 1$; the other points do not require specific analysis. Since $\cos(\pi/2) = 0$ and $\cos(\pi/3) = 1/2$, we have

$$\lim_{x\to 0^+} F_1(x) = \lim_{x\to 0^+} 2\cos\left(\frac{\pi}{2+x}\right) = 2\cos\left(\frac{\pi}{2}\right) = 0 = F_1(0)$$

and

$$\lim_{x\to 1^-} F_1(x) = \lim_{x\to 1^-} 2\cos\left(\frac{\pi}{2+x}\right) = 2\cos\left(\frac{\pi}{3}\right) = 1 = F_1(1).$$

Therefore, the function F_1 is continuous at $x = 0$ and $x = 1$, and thus on $\mathbb{R}$.

Let us analyze the increasingness of F_1. For any $x \in (0, 1)$, using standard differentiation rules, we obtain

$$F_1'(x) = \frac{2\pi}{(2+x)^2} \sin\left(\frac{\pi}{2+x}\right).$$

It is clear that $F_1'(x) \geq 0$. Therefore, F_1 is increasing on $(0, 1)$, and in fact on $\mathbb{R}$. Thanks to this, it is clear that, for any $x \in \mathbb{R}$, $F_1(x) \in [0, 1]$. This completes the proof. □

We denote the unit distribution defined by the CDF in Theorem 2.1 the cosine-ratio (CR) distribution. To the best of our knowledge, this characterizes a novel addition to the literature.

The PDF associated with the CR distribution is given by $f_1 = F_1'$ when F_1 is differentiable, that is,

$$f_1(x) = \frac{2\pi}{(2+x)^2} \sin\left(\frac{\pi}{2+x}\right), \quad x \in (0, 1),$$

which is completed with $f_1(x) = 0$ for any $x \notin (0, 1)$.

The QF associated with the CR distribution is given by $Q_1 = F_1^{-1}$. After some algebraic manipulations, we find that

$$Q_1(y) = \frac{\pi}{\arccos(y/2)} - 2, \quad y \in (0, 1).$$

Following the spirit in [3], the CR distribution can serve to generate distribution with various supports. For example, by composing it with the CDF of a Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$, i.e.,

$$G(x) = 1 - e^{-(x/\alpha)^\beta}, \quad x > 0,$$

which is completed with $G(x) = 0$ for $x \leq 0$, the function $U_1 = F_1(G)$ is a valid CDF with support $(0, +\infty)$, that is,

$$U_1(x) = 2\cos\left(\frac{\pi}{3 - e^{-(x/\alpha)^\beta}}\right), \quad x > 0,$$

which is completed with $U_1(x) = 0$ for $x \leq 0$. It can be interpreted as a modified Weibull distribution. This is just one example among an almost infinite number of such cases.

We end this subsection by a possible one-parameter extension of the CR distribution, defined by the following CDF:

$$F_1^\star(x) = \frac{1}{\cos(\pi/(2+\theta))} \cos\left(\frac{\pi}{2+\theta x}\right), \quad x \in (0,1),$$

with $\theta > 0$, which is completed with $F_1^\star(x) = 0$ for $x \le 0$ and $F_1^\star(x) = 1$ for $x \ge 1$.

These elements provide a sufficient theoretical foundation for the CR distribution to be studied in greater depth from a statistical perspective. We leave this aspect to the interested reader.

Complementing the CR distribution, and still based on the cosine function, one can show that the following function defines a valid CDF on the unit interval:

$$F_\dagger(x) = \frac{1}{2-\sqrt{2}} \left(2 - \frac{1}{\cos(\pi/(3+x))}\right), \quad x \in (0,1),$$

which is completed with $F_\dagger(x) = 0$ for $x \le 0$ and $F_\dagger(x) = 1$ for $x \ge 1$. We denote the unit distribution defined by this CDF the cosine-ratio 2 (CR2) distribution. To the best of our knowledge, this distribution is new to the literature. A detailed study of its properties is beyond the scope of this article.

2.2. **Second approach.** The second approach is based on the tangent function and relies on the CDF introduced in the theorem below.

Theorem 2.2. *The following function is a valid CDF of a unit distribution:*

$$F_2(x) = \frac{1}{1-\sqrt{5-2\sqrt{5}}} \left(1 - \tan\left(\frac{\pi}{4+x}\right)\right), \quad x \in (0,1),$$

which is completed with $F_2(x) = 0$ *for* $x \le 0$ *and* $F_2(x) = 1$ *for* $x \ge 1$.

Proof. Let us analyze the continuity of F_2 at $x = 0$ and $x = 1$; the other points do not require specific analysis. Since $\tan(\pi/4) = 1$ and $\tan(\pi/5) = \sqrt{5-2\sqrt{5}}$, we have

$$\lim_{x\to 0^+} F_2(x) = \lim_{x\to 0^+} \frac{1}{1-\sqrt{5-2\sqrt{5}}} \left(1 - \tan\left(\frac{\pi}{4+x}\right)\right)$$
$$= \frac{1}{1-\sqrt{5-2\sqrt{5}}} \left(1 - \tan\left(\frac{\pi}{4}\right)\right) = \frac{1}{1-\sqrt{5-2\sqrt{5}}} (1-1) = 0 = F_2(0)$$

and

$$\lim_{x\to 1^-} F_2(x) = \lim_{x\to 1^-} \frac{1}{1-\sqrt{5-2\sqrt{5}}}\left(1-\tan\left(\frac{\pi}{4+x}\right)\right)$$
$$= \frac{1}{1-\sqrt{5-2\sqrt{5}}}\left(1-\tan\left(\frac{\pi}{5}\right)\right) = \frac{1}{1-\sqrt{5-2\sqrt{5}}}\left(1-\sqrt{5-2\sqrt{5}}\right)$$
$$= 1 = F_2(1).$$

Therefore, the function F_2 is continuous at $x = 0$ and $x = 1$, and thus on $\mathbb{R}$.

Let us analyze the increasingness of F_2. For any $x \in (0, 1)$, using standard differentiation rules, we obtain

$$F_2'(x) = \frac{\pi}{1-\sqrt{5-2\sqrt{5}}}\frac{1}{(4+x)^2}\sec^2\left(\frac{\pi}{4+x}\right),$$

where $\sec(x) = 1/\cos(x)$. It is clear that $F_2'(x) \geq 0$. Therefore, F_2 is increasing on $(0, 1)$, and in fact on $\mathbb{R}$. Thanks to this, it is clear that, for any $x \in \mathbb{R}$, $F_2(x) \in [0, 1]$. This completes the proof. □

We denote the unit distribution defined by the CDF in Theorem 2.2 the tangent-ratio (TR) distribution. A review of existing literature suggests that this distribution has not been previously defined.

The PDF associated with the TR distribution is given by $f_2 = F_2'$ when F_2 is differentiable, that is,

$$f_2(x) = \frac{\pi}{1-\sqrt{5-2\sqrt{5}}}\frac{1}{(4+x)^2}\sec^2\left(\frac{\pi}{4+x}\right), \quad x \in (0, 1),$$

which is completed with $f_2(x) = 0$ for any $x \notin (0, 1)$.

The QF associated with the TR distribution is given by $Q_2 = F_2^{-1}$. After some algebraic manipulations, we find that

$$Q_2(y) = \frac{\pi}{\arctan\left(1-\left(1-\sqrt{5-2\sqrt{5}}\right)y\right)} - 4, \quad y \in (0, 1).$$

The TR distribution can serve to generate distribution with various supports. For example, by composing it with the CDF of a Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$ denoted G, $U_2 = F_2(G)$ is a valid CDF with support

$(0, +\infty)$, that is,

$$U_2(x) = \frac{1}{1 - \sqrt{5 - 2\sqrt{5}}}\left(1 - \tan\left(\frac{\pi}{5 - e^{-(x/\alpha)^\beta}}\right)\right), \quad x > 0,$$

which is completed with $U_2(x) = 0$ for $x \le 0$. It may be viewed as a modified Weibull distribution. This represents only one example among an essentially unbounded number of similar constructions.

We end this subsection by a possible one-parameter extension of the TR distribution, defined by the following CDF:

$$F_2^\star(x) = \frac{1}{1 - \tan(\pi/(4+\theta))}\left(1 - \tan\left(\frac{\pi}{4 + \theta x}\right)\right), \quad x \in (0,1),$$

with $\theta > 0$, which is completed with $F_2^\star(x) = 0$ for $x \le 0$ and $F_2^\star(x) = 1$ for $x \ge 1$.

These elements establish an adequate theoretical basis for a more detailed statistical investigation of the TR distribution. A deeper exploration of this topic is left to the interested reader.

2.3. **Third approach.** The third approach is based on the sine function and relies on the CDF introduced in the theorem below.

Theorem 2.3. *The following function is a valid CDF of a unit distribution:*

$$F_3(x) = \frac{2}{2 - \sqrt{3}}\left(1 - \sin\left(\frac{\pi}{2 + x}\right)\right), \quad x \in (0,1),$$

which is completed with $F_3(x) = 0$ *for* $x \le 0$ *and* $F_3(x) = 1$ *for* $x \ge 1$.

Proof. Let us analyze the continuity of F_3 at $x = 0$ and $x = 1$; the other points do not require specific analysis. Since $\sin(\pi/2) = 1$ and $\sin(\pi/3) = \sqrt{3}/2$, we have

$$\lim_{x \to 0^+} F_3(x) = \lim_{x \to 0^+} \frac{2}{2 - \sqrt{3}}\left(1 - \sin\left(\frac{\pi}{2 + x}\right)\right)$$
$$= \frac{2}{2 - \sqrt{3}}\left(1 - \sin\left(\frac{\pi}{2}\right)\right) = \frac{2}{2 - \sqrt{3}}(1 - 1) = 0 = F_3(0)$$

and

$$\lim_{x\to 1^-} F_3(x) = \lim_{x\to 1^-} \frac{2}{2-\sqrt{3}}\left(1-\sin\left(\frac{\pi}{2+x}\right)\right)$$
$$= \frac{2}{2-\sqrt{3}}\left(1-\sin\left(\frac{\pi}{3}\right)\right) = \frac{2}{2-\sqrt{3}}\left(1-\frac{\sqrt{3}}{2}\right) = 1 = F_3(1).$$

Therefore, the function F_3 is continuous at $x = 0$ and $x = 1$, and thus on $\mathbb{R}$.

Let us analyze the increasingness of F_3. For any $x \in (0, 1)$, using standard differentiation rules, we obtain

$$F_3'(x) = \frac{2\pi}{2-\sqrt{3}}\frac{1}{(2+x)^2}\cos\left(\frac{\pi}{2+x}\right).$$

It is clear that $F_3'(x) \geq 0$. Therefore, F_3 is increasing on $(0, 1)$, and in fact on $\mathbb{R}$. Thanks to this, it is clear that, for any $x \in \mathbb{R}$, $F_3(x) \in [0, 1]$. This completes the proof. □

We denote the unit distribution defined by the CDF in Theorem 2.3 the sine-ratio (SR) distribution. To the best of our knowledge, this represents a new contribution to the field.

The PDF associated with the SR distribution is given by $f_3 = F_3'$ when F_3 is differentiable, that is,

$$f_3(x) = \frac{2\pi}{2-\sqrt{3}}\frac{1}{(2+x)^2}\cos\left(\frac{\pi}{2+x}\right), \quad x \in (0, 1),$$

which is completed with $f_3(x) = 0$ for any $x \notin (0, 1)$.

The QF associated with the SR distribution is given by $Q_3 = F_3^{-1}$. After some algebraic manipulations, we find that

$$Q_3(y) = \frac{\pi}{\arcsin\left(1-\left(1-\sqrt{3}/2\right)y\right)} - 2, \quad y \in (0, 1).$$

The SR distribution can serve to generate distribution with various supports. For example, by composing it with the CDF of a Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$ denoted G, $U_3 = F_3(G)$ is a valid CDF with support $(0, +\infty)$, that is,

$$U_3(x) = \frac{2}{2-\sqrt{3}}\left(1-\sin\left(\frac{\pi}{3-e^{-(x/\alpha)^\beta}}\right)\right), \quad x \in (0, 1),$$

which is completed with $U_3(x) = 0$ for $x \leq 0$. It can be regarded as a modified Weibull distribution. This is just one example among an almost infinite number of such cases.

We end this subsection by a possible one-parameter extension of the SR distribution, defined by the following CDF:

$$F_3^\star(x) = \frac{1}{1 - \sin(\pi/(2+\theta))} \left(1 - \sin\left(\frac{\pi}{2+\theta x}\right)\right), \quad x \in (0,1),$$

with $\theta > 0$, which is completed with $F_3^\star(x) = 0$ for $x \leq 0$ and $F_3^\star(x) = 1$ for $x \geq 1$.

These elements offer a solid theoretical basis for a more detailed statistical study of the SR distribution. Further investigation is left to the interested reader.

3. Conclusion

In this article, we introduced new trigonometric unit distributions motivated by the limited availability of simple constructions in the recent literature. Three approaches based on the cosine, tangent, and sine functions were proposed, and their main distributional properties were derived. In particular, we provided explicit expressions for the CDF, PDF, and QF for each case. These results establish a tractable theoretical framework that may serve as a basis for further methodological developments and practical applications.

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