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# SOME NEW LOGARITHMIC TANGENT INTEGRAL FORMULAS

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ABSTRACT. This paper investigates integral formulas involving logarithmic and trigonometric functions, paying particular attention to the tangent function. We provide simplified proofs of selected classical formulas and derive several new ones, including those combining logarithmic, tangent and cotangent functions. All results are accompanied by complete proofs that highlight transparent methods which can be adapted to related problems.

## 1. INTRODUCTION

1.1. **Context.** Integral formulas play a central role in analysis, mathematical physics, probability theory, and number theory. Standard references such as [6] offer comprehensive compilations of classical results and remain indispensable to researchers. However, the search for more general integral formulas is ongoing, driven by the need to address increasingly complex problems in areas such as special functions, integral transforms, and applied mathematics. Recent developments, as reported in [1–5, 7–10], highlight substantial progress in extending classical results and establishing new integral formulas.

1.2. **Motivation.** The aim of this paper is to advance the study of integral formulas involving logarithmic and trigonometric functions, particularly the tangent function. We revisit selected formulas from [6], providing detailed and, in

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some cases, simple proofs. In addition, we derive several new integral formulas of this type, including ones that combine logarithmic, tangent, and cotangent functions. Complete proofs are given for all results, with an emphasis on methods that are both transparent and adaptable to related problems.

1.3. **Organization.** The paper is organized as follows: Section 2 presents a collection of simple integral formulas. Section 3 is devoted to more technical formulas. Concluding remarks are provided in Section 4.

## 2. Simple integral formulas

The formula presented below is general. It involves three functions: a logarithmic function, a trigonometric function and an intermediary function, denoted by $\phi$.

**Proposition 2.1.** *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an odd function. Then we have*

$$\int_0^{\pi/2} \phi\left(\ln(\tan(x))\right) dx = 0,$$

*provided that the integral exists.*

*Proof.* Using the Chasles integral relation and the change of variables $y = \pi/2 - x$, we obtain

$$\int_0^{\pi/2} \phi\left(\ln(\tan(x))\right) dx = \int_0^{\pi/4} \phi\left(\ln(\tan(x))\right) dx + \int_{\pi/4}^{\pi/2} \phi\left(\ln(\tan(x))\right) dx$$

$$= \int_0^{\pi/4} \phi\left(\ln(\tan(x))\right) dx + \int_{\pi/4}^{0} \phi\left(\ln\left(\tan\left(\frac{\pi}{2} - y\right)\right)\right)(-dy)$$

(2.1)

$$= \int_0^{\pi/4} \phi\left(\ln(\tan(x))\right) dx + \int_0^{\pi/4} \phi\left(\ln\left(\tan\left(\frac{\pi}{2} - y\right)\right)\right) dy.$$

Standard trigonometric formulas give

$$\tan\left(\frac{\pi}{2} - y\right) = \frac{\sin(\pi/2 - y)}{\cos(\pi/2 - y)} = \frac{\cos(y)}{\sin(y)} = \frac{1}{\tan(y)}.$$

Using this, a standard property of the logarithmic function, the fact that $\phi$ is odd and the existence of the involved integrals, we get

$$\begin{aligned}
&\int_0^{\pi/4} \phi\left(\ln(\tan(x))\right) dx + \int_0^{\pi/4} \phi\left(\ln\left(\tan\left(\frac{\pi}{2} - y\right)\right)\right) dy \\
&= \int_0^{\pi/4} \phi\left(\ln(\tan(x))\right) dx + \int_0^{\pi/4} \phi\left(\ln\left(\frac{1}{\tan(y)}\right)\right) dy \\
&= \int_0^{\pi/4} \phi\left(\ln(\tan(x))\right) dx + \int_0^{\pi/4} \phi\left(-\ln\left(\tan(y)\right)\right) dy \\
&= \int_0^{\pi/4} \phi\left(\ln(\tan(x))\right) dx - \int_0^{\pi/4} \phi\left(\ln\left(\tan(x)\right)\right) dx \\
&= 0. \qquad (2.2)
\end{aligned}$$

Combining Equations (2.1) and (2.2), we obtain

$$\int_0^{\pi/2} \phi\left(\ln(\tan(x))\right) dx = 0.$$

This concludes the proof. □

For any non-negative integer $n$, Proposition 2.1 applied to $\phi(x) = x^{2n+1}$ gives

$$\int_0^{\pi/2} \left(\ln(\tan(x))\right)^{2n+1} dx = 0.$$

This corresponds to [6, Entry 4.2276]. In particular, choosing $n = 0$, we find that

$$\int_0^{\pi/2} \ln(\tan(x))dx = 0. \qquad (2.3)$$

Another simple referenced formula is presented below.

**Proposition 2.2.** *For any $a > 0$, we have*

$$\int_0^{\pi/2} \ln(a\tan(x))dx = \frac{\pi}{2}\ln(a).$$

*This corresponds to [6, Entry 4.2273].*

*Proof.* Using a standard property of the logarithmic function and Equation (2.3), we immediately have

$$\int_0^{\pi/2} \ln(a\tan(x))dx = \int_0^{\pi/2} (\ln(a) + \ln(\tan(x)))\,dx$$
$$= \ln(a)\int_0^{\pi/2} dx + \int_0^{\pi/2} \ln(a\tan(x))dx$$
$$= \frac{\pi}{2}\ln(a) + 0 = \frac{\pi}{2}\ln(a).$$

This completes the proof. □

The result in [6, Entry 4.2273] becomes now totally transparent.

The proposition below is analogous to Proposition 2.1, but uses the cotangent function instead of the tangent function.

**Proposition 2.3.** *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an odd function. Then we have*

$$\int_0^{\pi/2} \phi\left(\ln(\mathrm{cotan}(x))\right) dx = 0,$$

*provided that the integral exists.*

*Proof.* Using $\mathrm{cotan}(x) = 1/\tan(x)$, a standard property of the logarithmic function, the fact that $\phi$ is odd and Proposition 2.1, we get

$$\int_0^{\pi/2} \phi\left(\ln(\mathrm{cotan}(x))\right) dx = \int_0^{\pi/2} \phi\left(\ln\left(\frac{1}{\tan(x)}\right)\right) dx$$
$$= \int_0^{\pi/2} \phi\left(-\ln\left(\tan(x)\right)\right) dx = -\int_0^{\pi/2} \phi\left(\ln\left(\tan(x)\right)\right) dx = 0.$$

This ends the proof.

As an alternative proof, applying the change of variables $x = \pi/2 - y$ in Proposition 2.1 yields the desired result. □

For any non-negative integer $n$, Proposition 2.3 applied to $\phi(x) = x^{2n+1}$ gives

$$\int_0^{\pi/2} \left(\ln(\mathrm{cotan}(x))\right)^{2n+1} dx = 0.$$

In particular, choosing $n = 0$, we find that

$$\int_0^{\pi/2} \ln(\mathrm{cotan}(x))dx = 0. \tag{2.4}$$

The proposition below is analogous to Proposition 2.2, but uses the cotangent function instead of the tangent function.

**Proposition 2.4.** *For any $a > 0$, we have*

$$\int_0^{\pi/2} \ln(a \operatorname{cotan}(x)) dx = \frac{\pi}{2} \ln(a).$$

*Proof.* Using a standard property of the logarithmic function and Equation (2.4), we immediately have

$$\int_0^{\pi/2} \ln(a \operatorname{cotan}(x)) dx = \int_0^{\pi/2} (\ln(a) + \ln(\operatorname{cotan}(x))) \, dx$$
$$= \ln(a) \int_0^{\pi/2} dx + \int_0^{\pi/2} \ln(a \operatorname{cotan}(x)) dx$$
$$= \frac{\pi}{2} \ln(a) + 0 = \frac{\pi}{2} \ln(a).$$

This concludes the proof.

As an alternative proof, applying the change of variables $x = \pi/2 - y$ in Proposition 2.2 yields the desired result. □

## 3. Technical integral formulas

The Proposition below considers an existing formula in [6] and provides a detailed proof.

**Proposition 3.1.** *We have*

$$\int_0^{\pi/2} \ln(\tan(x) + \operatorname{cotan}(x)) dx = \pi \ln(2).$$

*This corresponds to [6, Entry 4.22715].*

*Proof.* Using the definition of $\tan(x)$ and $\operatorname{cotan}(x)$ and the formula $\sin(2x) = 2 \sin(x) \cos(x)$, we have

$$\tan(x) + \operatorname{cotan}(x) = \frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)} = \frac{\sin^2(x) + \cos^2(x)}{\cos(x) \sin(x)}$$
$$= \frac{1}{\cos(x) \sin(x)} = \frac{2}{\sin(2x)}.$$

Combining this with a standard property of the logarithmic function gives

$$\ln(\tan(x) + \text{cotan}(x)) = \ln\left(\frac{2}{\sin(2x)}\right) = \ln(2) - \ln(\sin(2x)).$$

Using this and the change of variables $y = 2x$, we obtain

$$\int_0^{\pi/2} \ln(\tan(x) + \text{cotan}(x))dx = \int_0^{\pi/2} (\ln(2) - \ln(\sin(2x)))\, dx$$

$$= \ln(2)\int_0^{\pi/2} dx - \int_0^{\pi/2} \ln(\sin(2x))\, dx = \frac{\pi}{2}\ln(2) - \frac{1}{2}\int_0^{\pi} \ln(\sin(y))dy. \tag{3.1}$$

With reference to [6, Entry 4.2243], we have

$$\int_0^{\pi} \ln(\sin(y))dy = -\pi\ln(2). \tag{3.2}$$

It follows from Equations (3.1) and (3.2) that

$$\int_0^{\pi/2} \ln(\tan(x) + \text{cotan}(x))dx = \frac{\pi}{2}\ln(2) - \frac{1}{2}(-\pi\ln(2)) = \pi\ln(2).$$

This ends the proof. □

The proposition below presents a technical integral formula involving the cotangent function.

**Proposition 3.2.** *We have*

$$\int_0^{\pi/2} \ln(1 + \text{cotan}^2(x))dx = \pi\ln(2).$$

*Proof.* Using a standard property of the logarithmic function and Equation (2.3), we get

$$\int_0^{\pi/2} \ln(\tan(x) + \text{cotan}(x))dx = \int_0^{\pi/2} \ln\left(\tan(x)\left(1 + \frac{\text{cotan}(x)}{\tan(x)}\right)\right) dx$$

$$= \int_0^{\pi/2} \left(\ln(\tan(x)) + \ln\left(1 + \frac{\text{cotan}(x)}{\tan(x)}\right)\right) dx$$

$$= \int_0^{\pi/2} \ln(\tan(x))dx + \int_0^{\pi/2} \ln(1 + \text{cotan}^2(x))dx$$

$$= 0 + \int_0^{\pi/2} \ln(1 + \text{cotan}^2(x))dx = \int_0^{\pi/2} \ln(1 + \text{cotan}^2(x))dx.$$

It follows from Proposition 3.1 that

$$\int_0^{\pi/2} \ln(1+\text{cotan}^2(x))dx = \int_0^{\pi/2} \ln(\tan(x)+\text{cotan}(x))dx = \pi\ln(2).$$

This completes the proof. □

To the best of our knowledge, it is a new integral formula.

The tangent analogue is given in the proposition below. It is referenced in [6] under a specific configuration.

**Proposition 3.3.** *We have*

$$\int_0^{\pi/2} \ln(1+\tan^2(x))dx = \pi\ln(2).$$

*This corresponds to [6, Entry 4.22717 with $a = b = 1$].*

*Proof.* Using a standard property of the logarithmic function and Equation (2.4), we get

$$\int_0^{\pi/2} \ln(\tan(x)+\text{cotan}(x))dx = \int_0^{\pi/2} \ln\left(\text{cotan}(x)\left(1+\frac{\tan(x)}{\text{cotan}(x)}\right)\right)dx$$

$$= \int_0^{\pi/2} \left(\ln(\text{cotan}(x)) + \ln\left(1+\frac{\tan(x)}{\text{cotan}(x)}\right)\right)dx$$

$$= \int_0^{\pi/2} \ln(\text{cotan}(x))dx + \int_0^{\pi/2} \ln(1+\tan^2(x))dx$$

$$= 0 + \int_0^{\pi/2} \ln(1+\tan^2(x))dx = \int_0^{\pi/2} \ln(1+\tan^2(x))dx.$$

It follows from Proposition 3.1 that

$$\int_0^{\pi/2} \ln(1+\tan^2(x))dx = \int_0^{\pi/2} \ln(\tan(x)+\text{cotan}(x))dx = \pi\ln(2).$$

This concludes the proof.

As an alternative proof, applying the change of variables $x = \pi/2 - y$ in Proposition 3.2 yields the desired result. □

A technical and original integral formula is demonstrated below.

**Proposition 3.4.** *We have*

$$\int_0^{\pi/2} \ln\left(\frac{1+\operatorname{cotan}^2(x)}{1+\tan^2(x)}\right) dx = 0.$$

*Proof.* Using a standard property of the logarithmic function, and Propositions 3.2 and 3.3, we get

$$\begin{aligned}
&\int_0^{\pi/2} \ln\left(\frac{1+\operatorname{cotan}^2(x)}{1+\tan^2(x)}\right) dx \\
&= \int_0^{\pi/2} \left(\ln(1+\operatorname{cotan}^2(x)) - \ln(1+\tan^2(x))\right) dx \\
&= \int_0^{\pi/2} \ln(1+\operatorname{cotan}^2(x))dx - \int_0^{\pi/2} \ln(1+\tan^2(x))dx \\
&= \pi\ln(2) - \pi\ln(2) = 0.
\end{aligned}$$

This completes the proof. □

To the best of our knowledge, it is a new integral formula.

Another technical and original integral formula is demonstrated below.

**Proposition 3.5.** *We have*

$$\int_0^{\pi/2} \ln\left(2+\operatorname{cotan}^2(x)+\tan^2(x)\right) dx = 2\pi\ln(2).$$

*Proof.* Using a standard property of the logarithmic function, and Propositions 3.2 and 3.3, we get

$$\begin{aligned}
&\int_0^{\pi/2} \ln\left(2+\operatorname{cotan}^2(x)+\tan^2(x)\right) dx \\
&= \int_0^{\pi/2} \ln\left(1+\operatorname{cotan}^2(x)+\tan^2(x)+\operatorname{cotan}^2(x)\tan^2(x)\right) dx \\
&= \int_0^{\pi/2} \ln\left((1+\operatorname{cotan}^2(x))(1+\tan^2(x))\right) dx \\
&= \int_0^{\pi/2} \left(\ln(1+\operatorname{cotan}^2(x)) + \ln(1+\tan^2(x))\right) dx \\
&= \int_0^{\pi/2} \ln(1+\operatorname{cotan}^2(x))dx + \int_0^{\pi/2} \ln(1+\tan^2(x))dx \\
&= \pi\ln(2) + \pi\ln(2) = 2\pi\ln(2).
\end{aligned}$$

This ends the proof. □

To the best of our knowledge, it is a new integral formula.

## 4. Conclusion

In this paper, we revisited classical integral formulas involving logarithmic and trigonometric functions, establishing several new formulas, including combinations of logarithmic, tangent and cotangent functions. Future work could involve extending these formulas to more general classes of integrals and exploring their applications in special functions, complex analysis and mathematical physics.

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## References

[1] I. Ayoob: *On the evaluation of certain unsolved definite integrals*, European Journal of Pure and Applied Mathematics, **18**(3) (2025), 1–12.

[2] C. Chesneau: *On a new one-parameter arctangent-power integral*, International Journal of Open Problems in Computer Science and Mathematics, **17**(4) (2024), 1–8.

[3] C. Chesneau: *New integral formulas inspired by an old integral result*, International Journal of Open Problems in Computer Science and Mathematics, **18**(2) (2025), 53–71.

[4] C. Chesneau: *Some new integral formulas with applications*, International Journal of Open Problems in Computer Science and Mathematics, **18**(3) (2025), 2–22.

[5] C. Coine, C. Chesneau: *Solution to an open problem on a logarithmic integral and derived results*, Annals of West University of Timisoara - Mathematics and Computer Science, **61** (2025), 120–130.

[6] I.S. Gradshteyn, I.M. Ryzhik: *Table of Integrals, Series, and Products*, 7th Edition, Academic Press, (2007).

[7] R. Reynolds, A. Stauffer: *A definite integral involving the logarithmic function in terms of the Lerch function*, Mathematics, **7** (2019), 1–5.

[8] R. Reynolds, A. Stauffer: *Definite integral of arctangent and polylogarithmic functions expressed as a series*, Mathematics, **7** (2019), 1–7.

[9] R. Reynolds, A. Stauffer: *Derivation of logarithmic and logarithmic hyperbolic tangent integrals expressed in terms of special functions*, Mathematics, **8** (2020), 1–6.

[10] R. REYNOLDS, A. STAUFFER: *A quadruple definite integral expressed in terms of the Lerch function*, Symmetry, **13** (2021), 1–8.

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