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ON A HILBERT-TYPE INTEGRAL INEQUALITY WITH AN ORIGINAL NON-HOMOGENEOUS KERNEL

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ABSTRACT. In this paper, we derive a new Hilbert-type integral inequality involving a unique non-homogeneous kernel on the unit square. We provide a full self-contained proof of the main theorem.

1. INTRODUCTION

Hilbert-type integral inequalities serve as a cornerstone in mathematical analysis, functional analysis, and operator theory. They provide a framework for understanding the boundedness of integral operators across various function spaces. The classical Hilbert inequality is famously centered on the kernel

$$K(x,y)=\frac{1}{x+y},\quad x,y\in(0,+\infty).$$

It has inspired a vast literature on generalizations involving both homogeneous and non-homogeneous kernels. These extensions primarily focus on sharpening the associated constant factors or expanding the range of L^p spaces to which the inequalities apply. Such developments have been extensively documented in the literature (see, e.g., [1, 3–10]). Furthermore, the survey by [2] provides a comprehensive overview of these recent advancements.

Key words and phrases. Hilbert-type inequality; non-homogeneous kernel.
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In this paper, we focus on a novel kernel defined on the unit square $(0,1) \times (0,1)$, given by

$$K(x,y) = \frac{1}{\max(x,y) - xy}, \qquad x, y \in (0,1).$$

The structure of this kernel is particularly distinct; unlike traditional kernels that rely on simple addition or the absolute difference $|x-y|$, this formulation introduces a non-linear relationship through the maximum function and the product term. By using the specific properties of this kernel, we derive a new L^2-boundedness inequality.

The details are given in the next section. Section 3 provides a conclusion.

2. Result

The main theorem is stated below, followed by its detailed proof.

Theorem 2.1. *Let $f, g : (0,1) \to [0,+\infty)$ be measurable functions such that*

$$\int_0^1 f^2(x)dx < +\infty, \quad \int_0^1 g^2(y)dy < +\infty.$$

Then we have

$$\int_0^1 \int_0^1 \frac{f(x)g(y)}{\max(x,y) - xy} dxdy \leq 4 \left(\int_0^1 f^2(x)dx \right)^{1/2} \left(\int_0^1 g^2(y)dy \right)^{1/2}.$$

Proof. Let us set

$$I = \int_0^1 \int_0^1 \frac{f(x)g(y)}{\max(x,y) - xy} dxdy.$$

Then we can write

$$I = \int_0^1 \int_0^1 K(x,y) f(x) g(y) dxdy,$$

where

$$K(x,y) = \frac{1}{\max(x,y) - xy} = \begin{cases} \dfrac{1}{x(1-y)}, & y \leq x, \\ \dfrac{1}{y(1-x)}, & x \leq y. \end{cases}$$

Now, we rewrite the integral I as

$$I = \int_0^1 \int_0^1 K(x,y) \left(f(x) \frac{(x(1-x))^{1/4}}{(y(1-y))^{1/4}} \right) \left(g(y) \frac{(y(1-y))^{1/4}}{(x(1-x))^{1/4}} \right) dxdy.$$

The Cauchy-Schwarz integral inequality gives

$$I^2 \leq AB,$$

where

$$A = \int_0^1 \int_0^1 K(x,y) f^2(x) \frac{\sqrt{x(1-x)}}{\sqrt{y(1-y)}} dx dy$$

and

$$B = \int_0^1 \int_0^1 K(x,y) g^2(y) \frac{\sqrt{y(1-y)}}{\sqrt{x(1-x)}} dx dy.$$

By the Fubini-Tonelli integral theorem, we can write

$$A = \int_0^1 f^2(x) J(x) dx,$$

where

$$J(x) = \int_0^1 K(x,y) \frac{\sqrt{x(1-x)}}{\sqrt{y(1-y)}} dy.$$

Using the Chasles integral relation and the decomposition of the kernel, we have

$$\begin{aligned} J(x) &= \int_0^x K(x,y) \frac{\sqrt{x(1-x)}}{\sqrt{y(1-y)}} dy + \int_x^1 K(x,y) \frac{\sqrt{x(1-x)}}{\sqrt{y(1-y)}} dy \\ &= \frac{\sqrt{x(1-x)}}{x} \int_0^x \frac{dy}{\sqrt{y}(1-y)^{3/2}} + \frac{\sqrt{x(1-x)}}{1-x} \int_x^1 \frac{dy}{y^{3/2}\sqrt{1-y}}. \end{aligned}$$

For the first integral, let us consider the change of variables $y = \sin^2(\theta)$. Then, we have $dy = 2\sin(\theta)\cos(\theta)d\theta$, and $\sqrt{y} = \sin(\theta)$ and $1 - y = \cos^2(\theta)$. Hence, we derive

$$\int_0^x \frac{dy}{\sqrt{y}(1-y)^{3/2}} = 2\int_0^{\arcsin(\sqrt{x})} \sec^2(\theta) d\theta = 2\tan(\arcsin(\sqrt{x})) = 2\sqrt{\frac{x}{1-x}}.$$

Similarly, for the second integral, we obtain

$$\int_x^1 \frac{dy}{y^{3/2}\sqrt{1-y}} = 2\int_{\arcsin(\sqrt{x})}^{\pi/2} \csc^2(\theta) d\theta = 2\,\mathrm{cotan}(\arcsin(\sqrt{x})) = 2\sqrt{\frac{1-x}{x}}.$$

Therefore, we have

$$J(x) = \frac{\sqrt{x(1-x)}}{x} 2\sqrt{\frac{x}{1-x}} + \frac{\sqrt{x(1-x)}}{1-x} 2\sqrt{\frac{1-x}{x}} = 4.$$

Thus, we obtain

$$A = 4\int_0^1 f^2(x)dx.$$

Similarly, we get

$$B = 4\int_0^1 g^2(y)dy.$$

Consequently, we have

$$I^2 \leq 16\left(\int_0^1 f^2(x)dx\right)\left(\int_0^1 g^2(y)dy\right),$$

which implies

$$I \leq 4\left(\int_0^1 f^2(x)dx\right)^{1/2}\left(\int_0^1 g^2(y)dy\right)^{1/2}.$$

This proves the inequality, ending the proof. □

We emphasize that the inequality is L^2-bounded; other integral inequalities can also be derived, though they involve weighted integral norms of f and g.

3. Conclusion

In this paper, we established the Hilbert-type integral inequality

$$\int_0^1\int_0^1 \frac{f(x)g(y)}{\max(x,y)-xy}dxdy \leq 4\left(\int_0^1 f^2(x)dx\right)^{1/2}\left(\int_0^1 g^2(y)dy\right)^{1/2}.$$

Therefore, it is associated with the original nonhomogeneous kernel

$$K(x,y) = \frac{1}{\max(x,y)-xy}, \quad x,y \in (0,1).$$

To prove our main result, we employ a well-calibrated Cauchy-Schwarz integral inequality and various integral techniques. Potential extensions include proving that the constant 4 is the best possible, generalizing the inequality to L^p spaces, and considering the three-dimensional case through the kernel

$$K(x,y,z) = \frac{1}{\max(x,y,z)-xyz}, \quad x,y,z \in (0,1).$$

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