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ON A CONVEX INTEGRAL INEQUALITY OF PEDAGOGICAL INTEREST

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ABSTRACT. In this paper, we introduce a new convex integral inequality of pedagogical interest. We provide two distinct proofs and illustrate the underlying theory with several examples.

1. INTRODUCTION

Convex and concave functions play a vital role in mathematics and its applications. For completeness, we state their formal definitions below. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ with $b > a$.

Convex function: A function $f : [a, b] \to \mathbb{R}$ is said to be convex if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \tag{1.1}$$

Concave function: A function $f : [a, b] \to \mathbb{R}$ is said to be concave if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y).$$

One of the most significant implications of convex and concave functions is the establishment of a wide range of integral inequalities. Among these, the

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Jensen, Hermite-Hadamard and Fejér integral inequalities are the most prominent. Over the past few decades, substantial progress has been made in this field, with numerous researchers uncovering deeper connections between convexity, functional inequalities, and integral transforms. Comprehensive treatments and recent advances in this area can be found in a wealth of literature (see, e.g., [1–17]).

In this paper, we establish a new convex integral inequality. Because its proof relies entirely on elementary tools, it is of distinct pedagogical interest. We also provide an alternative proof utilizing the Hermite-Hadamard integral inequality. Our main result is stated and proved in the section below, alongside several concrete examples.

2. Result

Our main convex integral inequality is described in the theorem below.

Theorem 2.1. *Let $f : [0,1] \to \mathbb{R}$ be a convex integrable function. Then, we have*

$$\int_0^1 f(x)dx \geq \int_0^1 f\left(\frac{x}{2} + \frac{1}{4}\right) dx.$$

If the function f is concave rather than convex, then the inequality is reversed.

Proof. **Main proof.** Using the Chasles integral relation, we can write

$$\int_0^1 f(x)dx = \int_0^{1/2} f(x)dx + \int_{1/2}^1 f(x)dx.$$

Making the change of variables $x = u/2$, the first integral can be expressed as

$$\int_0^{1/2} f(x)dx = \frac{1}{2}\int_0^1 f\left(\frac{u}{2}\right) du.$$

Making the change of variables $x = (v+1)/2$, the second integral can be expressed as

$$\int_{1/2}^1 f(x)dx = \frac{1}{2}\int_0^1 f\left(\frac{v+1}{2}\right) dv.$$

Combining these equations and uniformizing the notations, we obtain

$$\int_0^1 f(x)dx = \frac{1}{2}\int_0^1 f\left(\frac{x}{2}\right)dx + \frac{1}{2}\int_0^1 f\left(\frac{x+1}{2}\right)dx$$
$$= \int_0^1 \left(\frac{1}{2}f\left(\frac{x}{2}\right) + \frac{1}{2}f\left(\frac{x+1}{2}\right)\right)dx.$$

Using the convexity of f, i.e., Equation (1.1) with $\lambda = 1/2$, we have

$$\frac{1}{2}f\left(\frac{x}{2}\right) + \frac{1}{2}f\left(\frac{x+1}{2}\right) \geq f\left(\frac{1}{2}\cdot\frac{x}{2} + \frac{1}{2}\cdot\frac{1+x}{2}\right)$$
$$= f\left(\frac{1+2x}{4}\right) = f\left(\frac{x}{2} + \frac{1}{4}\right). \tag{2.1}$$

Combining these equations, we get

$$\int_0^1 f(x)dx \geq \int_0^1 f\left(\frac{x}{2} + \frac{1}{4}\right)dx.$$

If f is concave, the inequality in Equation (2.1) is reversed, thereby reversing the final inequality. This concludes the main proof.

Another proof. An alternative proof utilizing the Hermite-Hadamard integral inequality as an intermediate result is detailed below. Using the convexity of f, i.e., Equation (1.1) with $\lambda = 1/2$, we have

$$f\left(\frac{x}{2} + \frac{1}{4}\right) = f\left(\frac{1}{2}\cdot x + \frac{1}{2}\cdot\frac{1}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f\left(\frac{1}{2}\right).$$

Integrating with respect to $x \in [0,1]$ yields

$$\int_0^1 f\left(\frac{x}{2} + \frac{1}{4}\right)dx \leq \int_0^1 \left(\frac{1}{2}f(x) + \frac{1}{2}f\left(\frac{1}{2}\right)\right)dx$$
$$= \frac{1}{2}\int_0^1 f(x) + \frac{1}{2}f\left(\frac{1}{2}\right)\int_0^1 dx$$
$$= \frac{1}{2}\int_0^1 f(x) + \frac{1}{2}f\left(\frac{1}{2}\right).$$

It follows from the Hermite-Hadamard integral inequality applied to the convex function f and the interval $[0,1]$ that

$$f\left(\frac{1}{2}\right) = f\left(\frac{0+1}{2}\right) \leq \frac{1}{1-0}\int_0^1 f(x)dx = \int_0^1 f(x)dx.$$

Combining these equations, we get

$$\int_0^1 f\left(\frac{x}{2}+\frac{1}{4}\right)dx \leq \frac{1}{2}\int_0^1 f(x) + \frac{1}{2}\int_0^1 f(x)dx = \int_0^1 f(x)dx.$$

This completes the alternative proof. □

We emphasize that the main proof relies solely on the elementary properties of convex functions. The alternative proof is also of technical interest; employing a similar approach yields the following generalization:

$$\int_0^1 f\left(\lambda x + (1-\lambda)\frac{1}{2}\right)dx \leq \int_0^1 f(x)dx,$$

for any $\lambda \in [0,1]$.

Some examples of applications of Theorem 2.1 are detailed below.

Example 1. *The function* $f(x) = \exp(x)$*,* $x \in [0,1]$*, is convex and integrable. We have*

$$\int_0^1 f(x)dx = \int_0^1 \exp(x)dx = \exp(1) - 1 \approx 1.7183$$

and

$$\int_0^1 f\left(\frac{x}{2}+\frac{1}{4}\right)dx = \int_0^1 \exp\left(\frac{x}{2}+\frac{1}{4}\right)dx = 2\left(\exp\left(\frac{1}{2}\right)-1\right)\exp\left(\frac{1}{4}\right)$$
$$\approx 1.6659.$$

Then, we have

$$\int_0^1 f(x)dx \approx 1.7183 \geq 1.6659 \approx \int_0^1 f\left(\frac{x}{2}+\frac{1}{4}\right)dx.$$

This illustrates Theorem 2.1.

Example 2. *The function* $f(x) = \exp(-x)$*,* $x \in [0,1]$*, is convex and integrable. We have*

$$\int_0^1 f(x)dx = \int_0^1 \exp(-x)dx = 1 - \exp(-1) \approx 0.6321$$

and

$$\int_0^1 f\left(\frac{x}{2}+\frac{1}{4}\right)dx = \int_0^1 \exp\left(-\frac{x}{2}-\frac{1}{4}\right)dx = 2\left(\exp\left(\frac{1}{2}\right)-1\right)\exp\left(-\frac{3}{4}\right)$$
$$\approx 0.6129.$$

Then, we have

$$\int_0^1 f(x)dx \approx 0.6321 \geq 0.6129 \approx \int_0^1 f\left(\frac{x}{2}+\frac{1}{4}\right)dx.$$

This illustrates Theorem 2.1.

Example 3. *The function* $f(x) = -\log(x)$, $x \in (0,1]$, *(the point* 0 *is removed without loss of generality) is convex and integrable. We have*

$$\int_0^1 f(x)dx = \int_0^1 \left(-\log(x)\right)dx = 1$$

and

$$\int_0^1 f\left(\frac{x}{2}+\frac{1}{4}\right)dx = \int_0^1 \left(-\log\left(\frac{x}{2}+\frac{1}{4}\right)\right)dx$$
$$= 1-\log(2)-\log\left(\frac{3\sqrt{3}}{8}\right) \approx 0.7384.$$

Then, we have

$$\int_0^1 f(x)dx = 1 \geq 0.7384 \approx \int_0^1 f\left(\frac{x}{2}+\frac{1}{4}\right)dx.$$

This illustrates Theorem 2.1.

Example 4. *The function* $f(x) = \arctan(x)$, $x \in [0,1]$, *is concave and integrable. We have*

$$\int_0^1 f(x)dx = \int_0^1 \arctan(x)dx = \frac{1}{4}(\pi - \log(4)) \approx 0.4388$$

and

$$\int_0^1 f\left(\frac{x}{2}+\frac{1}{4}\right)dx = \int_0^1 \arctan\left(\frac{x}{2}+\frac{1}{4}\right)dx$$
$$= \frac{1}{2}\left(-2\log\left(\frac{25}{17}\right) - \arctan\left(\frac{1}{4}\right) + 3\arctan\left(\frac{3}{4}\right)\right)$$
$$\approx 0.4571.$$

Then, we have

$$\int_0^1 f\left(\frac{x}{2}+\frac{1}{4}\right)dx \approx 0.4571 \geq 0.4388 \approx \int_0^1 f(x)dx.$$

This illustrates Theorem 2.1.

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