

STUDY OF A PARTICULAR HILBERT-TYPE INTEGRAL INEQUALITY

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ABSTRACT. In this paper, we investigate a particular Hilbert-type integral inequality involving a non-homogeneous kernel depending on an adjustable parameter. We establish the main inequality together with a complete and rigorous proof. Finally, some perspectives for future research are discussed.

1. INTRODUCTION

Hilbert-type integral inequalities are central to mathematical analysis. They provide sharp estimates for bilinear forms and boundedness criteria in functional analysis, operator theory, and harmonic analysis. The theory originated from the classical Hilbert integral inequality, which is associated with the kernel

$$K(x, y) = \frac{1}{x + y}, \quad x, y \in (0, \infty).$$

This result initiated extensive research into homogeneous and non-homogeneous kernels, weighted settings, and sharp constants (see, e.g., [1, 3–10]). For a comprehensive review of these modern developments, see [2].

Motivated by these advances, this paper introduces a new Hilbert-type integral inequality using the non-homogeneous kernel

$$K_\alpha(x, y) = \frac{1}{(xy - 1)^\alpha}, \quad x, y \in (1, \infty),$$

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where $\alpha > 1/2$ is an adjustable parameter. Unlike classical additive kernels, this formulation features a multiplicative structure with a singularity along the hyperbola $xy = 1$. A key novelty of our approach lies in the underlying domain $(1, \infty) \times (1, \infty)$, which represents a unconventional setting in the existing literature. Our proofs rely on the strategic application of the Cauchy-Schwarz integral inequality, weight function estimates, and integral substitutions.

The main theorem of this paper is presented in the following section, together with its complete proof and a remark.

2. THEOREM WITH PROOF

Our main integral inequality is described in the theorem below.

Theorem 2.1. *Let $\alpha > 1/2$, and $f, g : [1, \infty) \rightarrow [0, \infty)$ be measurable functions satisfying*

$$\int_1^\infty \frac{(x-1)^{1-\alpha}}{\sqrt{x}} f(x)^2 dx < \infty$$

and

$$\int_1^\infty \frac{(y-1)^{1-\alpha}}{\sqrt{y}} g(y)^2 dy < \infty.$$

Then, we have

$$\begin{aligned} & \int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(xy-1)^\alpha} dx dy \\ & \leq \mathcal{B}\left(\frac{1}{2}, \alpha - \frac{1}{2}\right) \left(\int_1^\infty \frac{(x-1)^{1-\alpha}}{\sqrt{x}} f(x)^2 dx\right)^{1/2} \left(\int_1^\infty \frac{(y-1)^{1-\alpha}}{\sqrt{y}} g(y)^2 dy\right)^{1/2}, \end{aligned}$$

where

$$\mathcal{B}(p, q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} dt$$

is the beta function.

Proof. For any $\alpha > 1/2$, let us consider

$$I_\alpha = \int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(xy-1)^\alpha} dx dy.$$

We insert the factor

$$\left(\frac{x-1}{y-1}\right)^{1/4} \left(\frac{y-1}{x-1}\right)^{1/4} = 1.$$

Hence, we have

$$I_\alpha = \int_1^\infty \int_1^\infty \left(\frac{f(x)}{(xy-1)^{\alpha/2}} \left(\frac{x-1}{y-1} \right)^{1/4} \right) \left(\frac{g(y)}{(xy-1)^{\alpha/2}} \left(\frac{y-1}{x-1} \right)^{1/4} \right) dx dy.$$

Applying the Cauchy–Schwarz integral inequality yields

$$I_\alpha^2 \leq AB,$$

where

$$A = \int_1^\infty \int_1^\infty \frac{f(x)^2}{(xy-1)^\alpha} \left(\frac{x-1}{y-1} \right)^{1/2} dx dy$$

and

$$B = \int_1^\infty \int_1^\infty \frac{g(y)^2}{(xy-1)^\alpha} \left(\frac{y-1}{x-1} \right)^{1/2} dx dy.$$

By the Fubini-Tonelli integral theorem, we have

$$A = \int_1^\infty f(x)^2 \omega_\alpha(x) dx,$$

where

$$\omega_\alpha(x) = \int_1^\infty \frac{1}{(xy-1)^\alpha} \left(\frac{x-1}{y-1} \right)^{1/2} dy.$$

Let us now perform the substitution

$$y = 1 + t, \quad t \in (0, \infty).$$

Then, we have

$$xy - 1 = x(1 + t) - 1 = (x - 1) + xt.$$

Therefore, we obtain

$$\begin{aligned} \omega_\alpha(x) &= \sqrt{x-1} \int_0^\infty \frac{dt}{\sqrt{t}[(x-1) + xt]^\alpha} \\ &= (x-1)^{1/2-\alpha} \int_0^\infty \frac{dt}{\sqrt{t}(1 + \lambda t)^\alpha}, \end{aligned}$$

where

$$\lambda = \frac{x}{x-1}.$$

Let us now perform the substitution

$$u = \lambda t.$$

Then, we have

$$dt = \frac{du}{\lambda}, \quad \sqrt{t} = \frac{\sqrt{u}}{\sqrt{\lambda}}.$$

Hence, we get

$$\begin{aligned} \omega_\alpha(x) &= (x-1)^{1/2-\alpha} \lambda^{-1/2} \int_0^\infty \frac{u^{-1/2}}{(1+u)^\alpha} du \\ &= \frac{(x-1)^{1-\alpha}}{\sqrt{x}} \int_0^\infty \frac{u^{-1/2}}{(1+u)^\alpha} du. \end{aligned}$$

Using the beta integral formula, we have

$$\begin{aligned} \int_0^\infty \frac{u^{-1/2}}{(1+u)^\alpha} du &= \int_0^\infty \frac{u^{1/2-1}}{(1+u)^{1/2+(\alpha-1/2)}} du \\ &= \mathcal{B}\left(\frac{1}{2}, \alpha - \frac{1}{2}\right), \end{aligned}$$

which is valid because $\alpha > 1/2$. We get

$$\omega_\alpha(x) = \mathcal{B}\left(\frac{1}{2}, \alpha - \frac{1}{2}\right) \frac{(x-1)^{1-\alpha}}{\sqrt{x}}.$$

Therefore, we obtain

$$A = \mathcal{B}\left(\frac{1}{2}, \alpha - \frac{1}{2}\right) \int_1^\infty \frac{(x-1)^{1-\alpha}}{\sqrt{x}} f(x)^2 dx.$$

Similarly, we have

$$B = \mathcal{B}\left(\frac{1}{2}, \alpha - \frac{1}{2}\right) \int_1^\infty \frac{(y-1)^{1-\alpha}}{\sqrt{y}} g(y)^2 dy.$$

Consequently, we have

$$I_\alpha^2 \leq \mathcal{B}\left(\frac{1}{2}, \alpha - \frac{1}{2}\right)^2 \left(\int_1^\infty \frac{(x-1)^{1-\alpha}}{\sqrt{x}} f(x)^2 dx \right) \left(\int_1^\infty \frac{(y-1)^{1-\alpha}}{\sqrt{y}} g(y)^2 dy \right).$$

Taking square roots yields

$$I_\alpha \leq \mathcal{B}\left(\frac{1}{2}, \alpha - \frac{1}{2}\right) \left(\int_1^\infty \frac{(x-1)^{1-\alpha}}{\sqrt{x}} f(x)^2 dx \right)^{1/2} \left(\int_1^\infty \frac{(y-1)^{1-\alpha}}{\sqrt{y}} g(y)^2 dy \right)^{1/2}.$$

This completes the proof. \square

Remark 2.1. When $\alpha = 1$ in Theorem 2.1, we recover the classical constant

$$\mathcal{B}\left(\frac{1}{2}, \frac{1}{2}\right) = \pi,$$

and thus obtain the elegant Hilbert-type integral inequality

$$\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{xy-1} dx dy \leq \pi \left(\int_1^\infty \frac{f(x)^2}{\sqrt{x}} dx \right)^{1/2} \left(\int_1^\infty \frac{g(y)^2}{\sqrt{y}} dy \right)^{1/2}.$$

3. CONCLUSION

In this paper, we introduced the Hilbert-type integral inequality

$$\begin{aligned} & \int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(xy-1)^\alpha} dx dy \\ & \leq \mathcal{B}\left(\frac{1}{2}, \alpha - \frac{1}{2}\right) \left(\int_1^\infty \frac{(x-1)^{1-\alpha}}{\sqrt{x}} f(x)^2 dx \right)^{1/2} \left(\int_1^\infty \frac{(y-1)^{1-\alpha}}{\sqrt{y}} g(y)^2 dy \right)^{1/2}, \end{aligned}$$

with $\alpha > 1/2$. Therefore, it is associated with the particular nonhomogeneous kernel

$$K_\alpha(x, y) = \frac{1}{(xy-1)^\alpha}, \quad x, y \in (1, \infty).$$

To establish our main result, we make use of a carefully calibrated application of the Cauchy-Schwarz integral inequality, combined with several auxiliary integral transformations and analytic techniques.

Possible extensions of this work include proving the optimality of the obtained constant factor, extending the inequality to L^p -spaces via appropriate interpolation methods, and investigating multidimensional analogues. In particular, a natural direction is the study of a three-dimensional generalization involving a kernel of the form

$$K_\alpha(x, y, z) = \frac{1}{(xyz-1)^\alpha}, \quad x, y, z \in (1, \infty),$$

which may lead to a new class of Hilbert-type integral inequalities in higher dimensions.

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REFERENCES

- [1] V. ADIYASUREN, T. BATBOLD, M. KRNIĆ: *Hilbert-type inequalities involving differential operators, the best constants and applications*, Math. Inequal. Appl., **18** (2015), 111–124.
- [2] Q. CHEN, B.C. YANG: *A survey on the study of Hilbert-type inequalities*, J. Inequal. Appl., **2015** (2015), 1–30.
- [3] C. CHESNEAU: *General inequalities of the Hilbert integral type using the method of switching to polar coordinates*, Hilbert J. Math. Anal., **3** (2024), 7–26.
- [4] C. CHESNEAU: *A three-parameter logarithmic generalization of the Hilbert integral inequality*, J. Math. Anal. Model., **6**(1) (2025), 68–81.
- [5] G.H. HARDY, J.E. LITTLEWOOD, G. PÓLYA: *Inequalities*, Cambridge University Press, Cambridge, (1934).
- [6] D.M. XIN: *A Hilbert-type integral inequality with the homogeneous kernel of zero degree*, Math. Theory Appl., **30**(2) (2010), 70–74.
- [7] J.S. XU: *Hardy-Hilbert's inequalities with two parameters*, Adv. Math., **36**(2) (2007), 63–76.
- [8] B.C. YANG: *On Hilbert's integral inequality*, J. Math. Anal. Appl., **220**(2) (1998), 778–785.
- [9] B.C. YANG: *Hilbert-Type Integral Inequalities*, Bentham Science Publishers, The United Arab Emirates, (2009).
- [10] B.C. YANG: *Hilbert-type integral inequality with non-homogeneous kernel*, J. Shanghai Univ., **17** (2011), 603–605.

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