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A HYPERBOLIC SINE VERSION OF THE HILBERT INTEGRAL INEQUALITY

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ABSTRACT. In this article, we establish the first hyperbolic sine version of the Hilbert integral inequality. In a sense, it extends a known sine-type Hilbert integral inequality to the hyperbolic setting. The proof relies on several changes of variables together with an intermediate Hilbert-type integral inequality on the unit square.

1. INTRODUCTION

The Hilbert integral inequality is a classical and influential result in analysis, particularly in the study of integral operators. One standard formulation is stated below. Let $f, g : (0,\infty) \to (0,\infty)$ be two measurable functions such that $\int_0^\infty f^2(x)dx < \infty$ and $\int_0^\infty g^2(y)dy < \infty$. Then, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \le \pi \left(\int_0^\infty f^2(x)dx\right)^{1/2} \left(\int_0^\infty g^2(y)dy\right)^{1/2}.$$

It has stimulated extensive research on both homogeneous and non-homogeneous kernel functions, weighted versions, multidimensional extensions, and the determination of best possible constant factors (see, for instance, [1, 3–12]). For comprehensive surveys and further developments concerning modern Hilbert-type integral inequalities, we refer the reader to [2].

Key words and phrases. Hilbert-type inequality; hyperbolic sine function; non-homogeneous kernel function.
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A Hilbert-type integral inequality involving the sine function was recently established in [5]. Its precise formulation is presented below. Let $f, g : (0, \pi/2) \to (0, \infty)$ be two measurable functions such that $\int_0^{\pi/2} f^2(x)dx < \infty$ and $\int_0^{\pi/2} g^2(y)dy < \infty$. Then, [5, Theorem 1] applied to $p = 2$ gives

$$\int_0^{\pi/2}\int_0^{\pi/2} \frac{f(x)g(y)}{\sin(x+y)}dxdy \leq \pi \left(\int_0^{\pi/2} f^2(x)dx\right)^{1/2} \left(\int_0^{\pi/2} g^2(y)dy\right)^{1/2}.$$

Based on this result, it is natural to address the following question: does a similar inequality remain valid when the sine function is replaced by its hyperbolic counterpart, namely the hyperbolic sine function

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad x \in \mathbb{R},$$

instead of $\sin(x)$?

This article answers this question in the affirmative. We establish a corresponding Hilbert-type integral inequality in the hyperbolic setting. We also provide a detailed proof that relies on several changes of variables and an intermediate Hilbert-type integral inequality on the unit square derived in [6], which is described below. Let $f, g : (0, 1) \to (0, \infty)$ be two measurable functions such that $\int_0^1 f^2(x)dx < \infty$ and $\int_0^1 g^2(y)dy < \infty$. Then, [6, Theorem 2.1] gives

$$\int_0^1\int_0^1 \frac{f(x)g(y)}{1-xy}dxdy \leq \pi \left(\int_0^1 f^2(x)dx\right)^{1/2} \left(\int_0^1 g^2(y)dy\right)^{1/2}. \tag{1.1}$$

Our main result and its proof are presented in the next section. A concluding discussion is given in Section 3.

2. Result

The proposed hyperbolic sine version of the Hilbert integral inequality is presented in the theorem below.

Theorem 2.1. *Let $f, g : (0, \infty) \to (0, \infty)$ be two measurable functions such that $\int_0^\infty f^2(x)dx < \infty$ and $\int_0^\infty g^2(y)dy < \infty$. Then, we have*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\sinh(x+y)}dxdy \leq \pi \left(\int_0^\infty f^2(x)dx\right)^{1/2} \left(\int_0^\infty g^2(y)dy\right)^{1/2}.$$

Proof. Let us define

$$I = \int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{\sinh(x+y)} dxdy.$$

We can write

$$\frac{1}{\sinh(x+y)} = \frac{2}{e^{x+y} - e^{-(x+y)}} = \frac{2e^{-(x+y)}}{1 - e^{-2(x+y)}}.$$

Hence, we have

$$I = 2 \int_0^{\infty} \int_0^{\infty} \frac{e^{-(x+y)}}{1 - e^{-2(x+y)}} f(x)g(y) dxdy.$$

Let us now make the changes of variables $u = e^{-2x}$ and $v = e^{-2y}$, so that

$$x = -\frac{1}{2}\ln(u), \quad y = -\frac{1}{2}\ln(v), \qquad dx = -\frac{du}{2u}, \quad dy = -\frac{dv}{2v}.$$

Then, we have

$$e^{-(x+y)} = (uv)^{1/2}, \qquad 1 - e^{-2(x+y)} = 1 - uv,$$

and thus

$$\frac{e^{-(x+y)}}{1 - e^{-2(x+y)}} dxdy = \frac{1}{4} \frac{dudv}{(1-uv)(uv)^{1/2}}.$$

Let us now define

$$F(u) = f\left(-\frac{1}{2}\ln(u)\right), \qquad G(v) = g\left(-\frac{1}{2}\ln(v)\right).$$

Then, we have

$$I = \frac{1}{2} \int_0^1 \int_0^1 \frac{F(u)G(v)}{(1-uv)(uv)^{1/2}} dudv.$$

Noting the separable structure in u and v, we now set

$$\Phi(u) = u^{-1/2}F(u), \qquad \Psi(v) = v^{-1/2}G(v).$$

Then, we have

$$I = \frac{1}{2} \int_0^1 \int_0^1 \frac{\Phi(u)\Psi(v)}{1-uv} dudv.$$

Applying [6, Theorem 2.1] to the functions Φ and Ψ, as recalled in Equation (1.1), we have

$$\int_0^1 \int_0^1 \frac{\Phi(u)\Psi(v)}{1-uv} dudv \leq \pi \left(\int_0^1 \Phi^2(u) du\right)^{1/2} \left(\int_0^1 \Psi^2(v) dv\right)^{1/2}.$$

Making the change of variables $u = e^{-2x}$, we obtain

$$\int_0^1 \Phi^2(u)du = \int_0^1 u^{-1}F^2(u)du = \int_0^1 u^{-1}f^2\left(-\frac{1}{2}\ln(u)\right)du$$
$$= \int_0^{\infty} e^{2x}f^2(x)\,2e^{-2x}dx = 2\int_0^{\infty} f^2(x)dx.$$

Similarly, with the change of variables $v = e^{-2y}$, we find that

$$\int_0^1 \Psi^2(v)dv = 2\int_0^{\infty} g^2(y)dy.$$

Finally, we have

$$I \leq \frac{\pi}{2}\left(2\int_0^{\infty} f^2(x)dx\right)^{1/2}\left(2\int_0^{\infty} g^2(y)dy\right)^{1/2}$$
$$= \pi\left(\int_0^{\infty} f^2(x)dx\right)^{1/2}\left(\int_0^{\infty} g^2(y)dy\right)^{1/2}.$$

This completes the proof of the theorem. □

3. Conclusion

In this article, we establish the first hyperbolic sine version of the Hilbert integral inequality, and provide a detailed proof of the result. A limitation of this work is that the optimality of the constant factor π is not addressed here. Future research may investigate this aspect further, as well as extensions to other hyperbolic functions, such as the hyperbolic cosine and hyperbolic tangent functions.

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