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ON A WEIGHTED HERMITE-HADAMARD-TYPE INTEGRAL INEQUALITY

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ABSTRACT. In this paper, we establish a new weighted Hermite-Hadamard-type integral inequality. The proof relies on an appropriate change of variables, the convexity of the underlying function, and an application of the Jensen integral inequality.

1. INTRODUCTION

The concept of convex functions is of central importance in various areas of mathematics, particularly in analysis, optimization theory, and geometry. It provides powerful tools for studying inequalities and structural properties of functions. This concept is recalled below. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ with $b > a$. A function $f : [a, b] \to \mathbb{R}$ is said to be convex if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1.1}$$

A profound implication of convex functions in mathematical analysis is the establishment of robust integral inequalities. The most celebrated among these is the Hermite-Hadamard (double) inequality, which states that the mean value of a convex function over a given interval is bounded below by the midpoint

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value of the function and above by the average of its boundary values. If $f : [a,b] \to \mathbb{R}$ is a convex function, then the Hermite-Hadamard integral inequality is expressed as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

In recent decades, substantial research has highlighted deep connections between convexity, functional inequalities, and integral transforms. These relationships have been extensively developed in the literature, where comprehensive frameworks and modern advances are presented (see, e.g., [1–17]).

In this paper, we contribute to this line of research by establishing a new weighted Hermite-Hadamard-type integral inequality. The weight function takes the form $(x-a)^\lambda$, where λ denotes an adjustable parameter such that $\lambda > -1$. If we take $\lambda = 0$, we rediscover the classical Hermite-Hadamard integral inequality. The proof relies on an appropriate change of variables, the convexity of the underlying function, and an application of the Jensen integral inequality, as recalled below. Let $g : [a,b] \to \mathbb{R}$ be an integrable function, and f be a convex function defined on the range of g. Then the Jensen integral inequality is expressed as follows:

$$f\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \leq \frac{1}{b-a}\int_a^b f(g(x))dx. \tag{1.3}$$

The main result is presented in the next section, together with its proof and three illustrative applications.

2. Result

Our new weighted Hermite-Hadamard-type integral inequality is described in the theorem below.

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $b > a$, $f : [a,b] \to \mathbb{R}$ be a convex function and $\lambda > -1$. Then, we have*

$$\frac{(b-a)^{\lambda+1}}{\lambda+1} f\left(a + \frac{\lambda+1}{\lambda+2}(b-a)\right) \leq \int_a^b (x-a)^\lambda f(x)dx$$
$$\leq \frac{(b-a)^{\lambda+1}}{(\lambda+1)(\lambda+2)}\Big(f(a) + (\lambda+1)f(b)\Big).$$

Proof. The proof is carried out in two steps: we first derive the lower bound, followed by the upper bound.

Proof of the lower bound. Let us consider the change of variables

$$x = a + (b-a)t^{1/(\lambda+1)}, \qquad t \in [0,1].$$

Then, we have

$$x - a = (b-a)t^{1/(\lambda+1)}$$

and

$$dx = \frac{b-a}{\lambda+1} t^{-\lambda/(\lambda+1)} dt.$$

Hence, we get

$$(x-a)^\lambda dx = (b-a)^\lambda t^{\lambda/(\lambda+1)} \cdot \frac{b-a}{\lambda+1} t^{-\lambda/(\lambda+1)} dt = \frac{(b-a)^{\lambda+1}}{\lambda+1} dt.$$

Therefore, we have

$$\int_a^b (x-a)^\lambda f(x) dx = \frac{(b-a)^{\lambda+1}}{\lambda+1} \int_0^1 f\big(a + (b-a)t^{1/(\lambda+1)}\big)\, dt.$$

Since f is convex, the Jensen integral inequality, as recalled in Equation (1.3), yields

$$\int_0^1 f\big(a + (b-a)t^{1/(\lambda+1)}\big)\, dt \geq f\left(\int_0^1 \Big(a + (b-a)t^{1/(\lambda+1)}\Big) dt\right).$$

Evaluating the integral, we get

$$\int_0^1 t^{1/(\lambda+1)} dt = \left[\frac{\lambda+1}{\lambda+2} t^{(\lambda+2)/(\lambda+1)}\right]_0^1 = \frac{\lambda+1}{\lambda+2},$$

which gives

$$\int_0^1 \Big(a + (b-a)t^{1/(\lambda+1)}\Big) dt = a \int_0^1 dt + (b-a) \int_0^1 t^{1/(\lambda+1)} dt = a + \frac{\lambda+1}{\lambda+2}(b-a).$$

Therefore, we have

$$\int_0^1 f\big(a + (b-a)t^{1/(\lambda+1)}\big)\, dt \geq f\left(a + \frac{\lambda+1}{\lambda+2}(b-a)\right).$$

Multiplying by

$$\frac{(b-a)^{\lambda+1}}{\lambda+1}$$

yields

$$\int_a^b (x-a)^\lambda f(x)dx \geq \frac{(b-a)^{\lambda+1}}{\lambda+1} f\left(a + \frac{\lambda+1}{\lambda+2}(b-a)\right).$$

This completes the proof of the lower bound.

Proof of the upper bound. The first step is carried out in a manner similar to that used in the proof of the lower bound. Let us define the change of variables

$$x = a + (b-a)t^{1/(\lambda+1)}, \qquad t \in [0,1].$$

Then, we have

$$x - a = (b-a)t^{1/(\lambda+1)}$$

and

$$dx = \frac{b-a}{\lambda+1} t^{-\lambda/(\lambda+1)} dt.$$

Hence, we get

$$(x-a)^\lambda dx = (b-a)^\lambda t^{\lambda/(\lambda+1)} \cdot \frac{b-a}{\lambda+1} t^{-\lambda/(\lambda+1)} dt = \frac{(b-a)^{\lambda+1}}{\lambda+1} dt.$$

Therefore, we obtain

$$\int_a^b (x-a)^\lambda f(x)dx = \frac{(b-a)^{\lambda+1}}{\lambda+1} \int_0^1 f\big(a + (b-a)t^{1/(\lambda+1)}\big)\, dt.$$

Since f is convex, by Equation (1.1), we have

$$f\big((1-s)a + sb\big) \leq (1-s)f(a) + sf(b), \qquad s \in [0,1].$$

If we take

$$s = t^{1/(\lambda+1)},$$

then we obtain

$$f\big(a + (b-a)t^{1/(\lambda+1)}\big) = f\Big(\big(1 - t^{1/(\lambda+1)}\big)\, a + t^{1/(\lambda+1)} b\Big)$$
$$\leq \left(1 - t^{1/(\lambda+1)}\right) f(a) + t^{1/(\lambda+1)} f(b).$$

Integrating over $[0,1]$ gives

$$\int_0^1 f\big(a + (b-a)t^{1/(\lambda+1)}\big)\, dt \leq f(a) \int_0^1 \left(1 - t^{1/(\lambda+1)}\right) dt + f(b) \int_0^1 t^{1/(\lambda+1)} dt.$$

We have

$$\int_0^1 t^{1/(\lambda+1)}dt = \left[\frac{\lambda+1}{\lambda+2}t^{(\lambda+2)/(\lambda+1)}\right]_0^1 = \frac{\lambda+1}{\lambda+2}$$

and

$$\int_0^1 \left(1 - t^{1/(\lambda+1)}\right)dt = 1 - \int_0^1 t^{1/(\lambda+1)}dt = 1 - \frac{\lambda+1}{\lambda+2} = \frac{1}{\lambda+2}.$$

Thus, we obtain

$$\int_0^1 f\left(a + (b-a)t^{1/(\lambda+1)}\right) dt \leq \frac{f(a)}{\lambda+2} + \frac{\lambda+1}{\lambda+2}f(b).$$

Multiplying by

$$\frac{(b-a)^{\lambda+1}}{\lambda+1}$$

yields

$$\int_a^b (x-a)^{\lambda} f(x)dx \leq \frac{(b-a)^{\lambda+1}}{(\lambda+1)(\lambda+2)}\Big(f(a) + (\lambda+1)f(b)\Big).$$

The proof of the upper bound is complete. This ends the proof of the theorem. □

In particular, if we take $\lambda = 0$, then Theorem 2.1 yields

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2},$$

so

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

which is the Hermite-Hadamard integral inequality, as recalled in Equation (1.2).

We also note that λ may be negative, which opens up new directions for research.

In the case where f is concave instead of convex, the inequality in Theorem 2.1 is reversed.

To evaluate the sharpness of the inequality, three simple examples of applications of Theorem 2.1 are detailed below.

Example 1. *The function* $f(x) = x^2$*,* $x \in [0, 1]$*, is convex. If we take* $a = 0$*,* $b = 1$ *and* $\lambda = 6$*, then we have*

$$\int_a^b (x-a)^\lambda f(x)dx = \int_0^1 x^6 \cdot x^2 dx = \frac{1}{9} \approx 0.1111,$$

$$\frac{(b-a)^{\lambda+1}}{\lambda+1} f\left(a + \frac{\lambda+1}{\lambda+2}(b-a)\right) = \frac{1}{7} f\left(\frac{7}{8}\right) = \frac{1}{7}\left(\frac{7}{8}\right)^2 \approx 0.1093$$

and

$$\frac{(b-a)^{\lambda+1}}{(\lambda+1)(\lambda+2)}\Big(f(a) + (\lambda+1)f(b)\Big) = \frac{1}{7 \cdot 8}(0+7) = \frac{1}{8} = 0.125.$$

Therefore, we have

$$\frac{(b-a)^{\lambda+1}}{\lambda+1} f\left(a + \frac{\lambda+1}{\lambda+2}(b-a)\right) \leq \int_a^b (x-a)^\lambda f(x)dx \leq \frac{(b-a)^{\lambda+1}}{(\lambda+1)(\lambda+2)}\Big(f(a) + (\lambda+1)f(b)\Big).$$

Theorem 2.1 is thus illustrated.

Example 2. *The function* $f(x) = \exp(x)$*,* $x \in [0, 1]$*, is convex. If we take* $a = 0$*,* $b = 1$ *and* $\lambda = 2$*, then we have*

$$\int_a^b (x-a)^\lambda f(x)dx = \int_0^1 x^2 \exp(x)dx = \exp(1) - 2 \approx 0.7182,$$

$$\frac{(b-a)^{\lambda+1}}{\lambda+1} f\left(a + \frac{\lambda+1}{\lambda+2}(b-a)\right) = \frac{1}{3} f\left(\frac{3}{4}\right) = \frac{1}{3}\exp\left(\frac{3}{4}\right) \approx 0.7056$$

and

$$\frac{(b-a)^{\lambda+1}}{(\lambda+1)(\lambda+2)}\Big(f(a) + (\lambda+1)f(b)\Big) = \frac{1}{3 \cdot 4}(1 + 3\exp(1)) \approx 0.7629.$$

Therefore, we have

$$\frac{(b-a)^{\lambda+1}}{\lambda+1} f\left(a+\frac{\lambda+1}{\lambda+2}(b-a)\right) \leq \int_a^b (x-a)^\lambda f(x)dx$$
$$\leq \frac{(b-a)^{\lambda+1}}{(\lambda+1)(\lambda+2)}\Big(f(a)+(\lambda+1)f(b)\Big).$$

Theorem 2.1 is thereby verified.

Example 3. *The function* $f(x) = \exp(-x)$, $x \in [0,1]$*, is convex. If we take* $a = 0$, $b = 1$ *and* $\lambda = 4$*, then we have*

$$\int_a^b (x-a)^\lambda f(x)dx = \int_0^1 x^4 \exp(-x)dx = 24 - 65\exp(-1) \approx 0.0878,$$

$$\frac{(b-a)^{\lambda+1}}{\lambda+1} f\left(a+\frac{\lambda+1}{\lambda+2}(b-a)\right) = \frac{1}{5} f\left(\frac{5}{6}\right) = \frac{1}{5}\exp\left(-\frac{5}{6}\right)$$
$$\approx 0.0869$$

and

$$\frac{(b-a)^{\lambda+1}}{(\lambda+1)(\lambda+2)}\Big(f(a)+(\lambda+1)f(b)\Big) = \frac{1}{5\cdot 6}\left(1+5\exp(-1)\right)$$
$$\approx 0.0946.$$

Therefore, we have

$$\frac{(b-a)^{\lambda+1}}{\lambda+1} f\left(a+\frac{\lambda+1}{\lambda+2}(b-a)\right) \leq \int_a^b (x-a)^\lambda f(x)dx$$
$$\leq \frac{(b-a)^{\lambda+1}}{(\lambda+1)(\lambda+2)}\Big(f(a)+(\lambda+1)f(b)\Big).$$

This example confirms Theorem 2.1.

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