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CORRECTIONS TO TWO HARDY-HILBERT-TYPE INTEGRAL INEQUALITIES

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ABSTRACT. In this paper, we present corrections to two Hardy-Hilbert-type integral inequalities that were recently established in the literature. These corrections primarily concern the definition of the weighted integral norms of the functions appearing in the corresponding upper bounds. By reformulating these norms appropriately, we obtain corrected versions of the inequalities and clarify the conditions under which the results hold.

1. INTRODUCTION

In 2004, B.C. Yang [3] established an important Hardy-Hilbert-type integral inequality depending on one adjustable parameter, λ, and two particular pairs of conjugate exponents, (p, q) and (r, s). A formal statement of this result is given below. Let $p, q, r, s > 1$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad \frac{1}{r} + \frac{1}{s} = 1.$$

Let $\lambda > 0$ and $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that

$$\int_0^{+\infty} x^{p(1-\lambda/r)-1} f^p(x) dx < +\infty, \quad \int_0^{+\infty} y^{q(1-\lambda/s)-1} g^q(y) dy < +\infty.$$

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Then, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x^{\lambda}+y^{\lambda}}dxdy \leq \frac{\pi}{\lambda\sin(\pi/r)} \cdot \left(\int_0^{+\infty} x^{p(1-\lambda/r)-1}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty} y^{q(1-\lambda/s)-1}g^q(y)dy\right)^{1/q}, \tag{1.1}$$

where the constant factor $\pi/(\lambda\sin(\pi/r))$ is optimal. See also [1, Equation (89)].

Building on this result, it appears that two corrections to secondary results in [2] are required. The purpose of the next section is to present and justify these corrections.

2. Corrections

The first correction needs to be applied to [2, Equation (1.3)], as described below.

Proposition 2.1. *Let $p, q > 1$ be such that*

$$\frac{1}{p}+\frac{1}{q}=1.$$

Let $\alpha > -1$ and $f, g : [0,+\infty) \to [0,+\infty)$ be two functions such that

$$\int_0^{+\infty} x^{(1-\alpha)p/2-1}f^p(x)dx < +\infty, \quad \int_0^{+\infty} y^{(1-\alpha)q/2-1}g^q(y)dy < +\infty.$$

Then, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x^{\alpha+1}+y^{\alpha+1}}dxdy \leq \frac{\pi}{\alpha+1}\left(\int_0^{+\infty} x^{(1-\alpha)p/2-1}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty} y^{(1-\alpha)q/2-1}g^q(y)dy\right)^{1/q},$$

where the constant factor $\pi/(\alpha+1)$ is optimal.

Proof. It suffices to apply Equation (1.1) with the following configuration: $r = s = 2$, $\lambda = \alpha+1$, and to note that $1-(\alpha+1)/2 = (1-\alpha)/2$ and

$$\frac{\pi}{\lambda\sin(\pi/r)} = \frac{\pi}{(\alpha+1)\sin(\pi/2)} = \frac{\pi}{\alpha+1}.$$

. ∎

Compared with [2, Equation (1.3)], the weight functions appearing in the integral norms have been modified and now depend explicitly on α. In particular, the factor $1-\alpha$ appears to be missing twice in [2, Equation (1.3)].

This correction has an immediate consequence for [2, Proposition 3.5], which must now be reformulated as follows.

Proposition 2.2. *Let* $p, q > 1$ *be such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let $\alpha > -1/2$ *and* $f, g : [0, +\infty) \to [0, +\infty)$ *be two functions such that*

$$\int_0^{+\infty} x^{p/2-1} f^p(x)dx < +\infty, \quad \int_0^{+\infty} y^{q/2-1} g^q(y)dy < +\infty$$

and

$$\int_0^{+\infty} x^{(1-\alpha)p/2-1} f^p(x)dx < +\infty, \quad \int_0^{+\infty} y^{(1-\alpha)q/2-1} g^q(y)dy < +\infty.$$

Let $\beta \in [0, 1]$. *Then, we have*

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{(x^\alpha + y^\alpha)^\beta}{x^{\alpha+1} + y^{\alpha+1}} f(x)g(y)dxdy \\
&\leq \Upsilon_{\alpha,\beta} \left(\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right)^{\beta/p} \left(\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right)^{\beta/q} \\
&\times \left(\int_0^{+\infty} x^{(1-\alpha)p/2-1} f^p(x)dx\right)^{(1-\beta)/p} \left(\int_0^{+\infty} y^{(1-\alpha)q/2-1} g^q(y)dy\right)^{(1-\beta)/q},
\end{aligned}$$

where

$$\Upsilon_{\alpha,\beta} = \frac{\pi}{\alpha+1} 2^\beta \left(\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right)^{-\beta}. \tag{2.1}$$

Proof. The first steps of the proof follow those in [2, Proof of Proposition 3.5]. Introducing the exponent β, we can write

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{(x^\alpha + y^\alpha)^\beta}{x^{\alpha+1} + y^{\alpha+1}} f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} \left(\frac{x^\alpha + y^\alpha}{x^{\alpha+1} + y^{\alpha+1}} f(x)g(y)\right)^\beta \left(\frac{1}{x^{\alpha+1} + y^{\alpha+1}} f(x)g(y)\right)^{1-\beta} dxdy.
\end{aligned}$$

The Hölder integral inequality gives

$$\int_0^{+\infty}\int_0^{+\infty}\left(\frac{x^\alpha+y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)\right)^\beta\left(\frac{1}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)\right)^{1-\beta}dxdy$$
$$\leq\left(\int_0^{+\infty}\int_0^{+\infty}\frac{x^\alpha+y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy\right)^\beta$$
$$\times\left(\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy\right)^{1-\beta}. \tag{2.2}$$

[2, Proposition 3.3] ensures that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^\alpha+y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy$$
$$\leq\Lambda_\alpha\left(\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right)^{1/q}, \tag{2.3}$$

where

$$\Lambda_\alpha=\frac{2\pi}{\alpha+1}\left(\sin\left(\frac{\pi}{2(\alpha+1)}\right)\right)^{-1}.$$

On the other hand, the newly corrected Proposition 2.1 ensures that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)dxdy$$
$$\leq\frac{\pi}{\alpha+1}\left(\int_0^{+\infty}x^{(1-\alpha)p/2-1}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(1-\alpha)q/2-1}g^q(y)dy\right)^{1/q}. \tag{2.4}$$

Combining Equations (2.2), (2.3) and (2.4), and using the identity $1/p+1/q=1$, we have

$$\int_0^{+\infty}\int_0^{+\infty}\left(\frac{x^\alpha+y^\alpha}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)\right)^\beta\left(\frac{1}{x^{\alpha+1}+y^{\alpha+1}}f(x)g(y)\right)^{1-\beta}dxdy$$
$$\leq\left(\Lambda_\alpha\left(\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right)^{1/q}\right)^\beta$$
$$\times\left(\frac{\pi}{\alpha+1}\left(\int_0^{+\infty}x^{(1-\alpha)p/2-1}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(1-\alpha)q/2-1}g^q(y)dy\right)^{1/q}\right)^{1-\beta}$$
$$=\Lambda_\alpha^\beta\left(\frac{\pi}{\alpha+1}\right)^{1-\beta}\left(\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right)^{\beta/p}\left(\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right)^{\beta/q}$$

$$\times \left(\int_0^{+\infty} x^{(1-\alpha)p/2-1} f^p(x)dx \right)^{(1-\beta)/p} \left(\int_0^{+\infty} y^{(1-\alpha)q/2-1} g^q(y)dy \right)^{(1-\beta)/q}$$
$$= \Upsilon_{\alpha,\beta} \left(\int_0^{+\infty} x^{p/2-1} f^p(x)dx \right)^{\beta/p} \left(\int_0^{+\infty} y^{q/2-1} g^q(y)dy \right)^{\beta/q}$$
$$\times \left(\int_0^{+\infty} x^{(1-\alpha)p/2-1} f^p(x)dx \right)^{(1-\beta)/p} \left(\int_0^{+\infty} y^{(1-\alpha)q/2-1} g^q(y)dy \right)^{(1-\beta)/q},$$

where

$$\Upsilon_{\alpha,\beta} = \Lambda_\alpha^\beta \left(\frac{\pi}{\alpha+1} \right)^{1-\beta} = \frac{\pi}{\alpha+1} 2^\beta \left(\sin\left(\frac{\pi}{2(\alpha+1)} \right) \right)^{-\beta},$$

as indicated in Equation (2.1). This concludes the proof of Proposition 2.2. □

Compared with [2, Proposition 3.5], the role of the parameter β is notably more complex; it modulates different weighted integral norms of f and g in a manner that also depends on α.

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