J COMP SCI
APPL MATH
Journal of Computer Science and Applied Mathematics
Vol. 8, no.2, (2026), 207–212
ISSN: 1857-9582 https://doi.org/10.37418/jcsam.8.2.3

ON THREE ONE-PARAMETER FAMILIES OF CYCLIC INEQUALITIES

Christophe Chesneau

ABSTRACT. This paper is devoted to the study of three one-parameter families of cyclic inequalities. Complete proofs of these results are provided.

1. INTRODUCTION

Cyclic inequalities have numerous applications in the study and proof of fractional inequalities. One of the most classical and well-known examples is the Nesbitt inequality, stated below. For any $a, b, c > 0$, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

This inequality was generalized by introducing an adjustable parameter, θ, in [6]. The resulting generalization is stated below. For any $a, b, c, \theta > 0$, we have

$$\frac{a}{\theta b+c} + \frac{b}{\theta c+a} + \frac{c}{\theta a+b} \geq \frac{3}{\theta+1}.$$

This inequality gives rise to a one-parameter family of cyclic inequalities, with the classical Nesbitt inequality recovered as the special case $\theta = 1$. Various proofs, extensions, refinements, and related results concerning this inequality have been established; see, for example, [1–6].

Key words and phrases. cyclic inequalities, Nesbitt inequality.
Submitted: 13.06.2026; *Accepted:* 30.06.2026; *Published:* 17.07.2026.

In this paper, following the approach of [6], we introduce and establish three one-parameter families of cyclic inequalities of increasing mathematical complexity. In a certain sense, these results complete the collection of cyclic inequalities presented in [3].

2. Cyclic inequalities

Our first one-parameter family of cyclic inequalities is stated in the theorem below.

Theorem 2.1. *For any* $a, b, c > 0$ *and* $\theta > -2$*, we have*

$$\frac{a^2(a+b\theta-b)}{a^2+ab\theta+b^2}+\frac{b^2(b+c\theta-c)}{b^2+bc\theta+c^2}+\frac{c^2(c+a\theta-a)}{c^2+ca\theta+a^2}\geq\frac{\theta}{\theta+2}(a+b+c).$$

Proof. Using a non-trivial decomposition and the inequality $x+1/x\geq 2$ for any $x>0$, we have

$$\frac{a^2(a+b\theta-b)}{a^2+ab\theta+b^2}=a-\frac{a+b}{\theta+a/b+b/a}\geq a-\frac{a+b}{\theta+2}.$$

Similarly, we have

$$\frac{b^2(b+c\theta-c)}{b^2+bc\theta+c^2}=b-\frac{b+c}{\theta+b/c+c/b}\geq b-\frac{b+c}{\theta+2}$$

and

$$\frac{c^2(c+a\theta-a)}{c^2+ca\theta+a^2}=c-\frac{c+a}{\theta+c/a+a/c}\geq c-\frac{c+a}{\theta+2}.$$

Adding the three inequalities above, we get

$$\frac{a^2(a+b\theta-b)}{a^2+ab\theta+b^2}+\frac{b^2(b+c\theta-c)}{b^2+bc\theta+c^2}+\frac{c^2(c+a\theta-a)}{c^2+ca\theta+a^2}$$
$$\geq a+b+c-2\cdot\frac{a+b+c}{\theta+2}=\frac{\theta}{\theta+2}(a+b+c).$$

This concludes the proof. □

Some special cases of Theorem 2.1 are presented below.

If we take $\theta=-1$, then we have

$$\frac{a^2(a-2b)}{a^2-ab+b^2}+\frac{b^2(b-2c)}{b^2-bc+c^2}+\frac{c^2(c-2a)}{c^2-ca+a^2}\geq-(a+b+c).$$

If we take $\theta = 0$, then we have

$$\frac{a^2(a-b)}{a^2+b^2} + \frac{b^2(b-c)}{b^2+c^2} + \frac{c^2(c-a)}{c^2+a^2} \geq 0,$$

which is not trivial at first glance.

If we take $\theta = 1$, then we have

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{1}{3}(a+b+c).$$

If we take $\theta = 2$, then we have

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \geq \frac{1}{2}(a+b+c).$$

Naturally, other integer or decimal values of θ can be handled in a similar fashion.

Our second one-parameter family of cyclic inequalities is stated in the theorem below.

Theorem 2.2. *For any* $a, b, c > 0$ *and* $\theta > -2$*, we have*

$$\frac{a^3+a^2b\theta+a^2b+ab^2\theta-b^3}{a^2+ab\theta+2ab+b^2\theta+2b^2} + \frac{b^3+b^2c\theta+b^2c+bc^2\theta-c^3}{b^2+bc\theta+2bc+c^2\theta+2c^2}$$
$$+ \frac{c^3+c^2a\theta+c^2a+ca^2\theta-a^3}{c^2+ca\theta+2ca+a^2\theta+2a^2} \geq \frac{\theta}{\theta+2}(a+b+c).$$

Proof. Using a non-trivial decomposition and the inequality $x + 1/x \geq 2$ for any $x > 0$, we have

$$\frac{a^3+a^2b\theta+a^2b+ab^2\theta-b^3}{a^2+ab\theta+2ab+b^2\theta+2b^2} = a - \frac{a+b}{\theta+(a+b)/b+b/(a+b)} \geq a - \frac{a+b}{\theta+2}.$$

Similarly, we have

$$\frac{b^3+b^2c\theta+b^2c+bc^2\theta-c^3}{b^2+bc\theta+2bc+c^2\theta+2c^2} = b - \frac{b+c}{\theta+(b+c)/c+c/(b+c)} \geq b - \frac{b+c}{\theta+2}$$

and

$$\frac{c^3+c^2a\theta+c^2a+ca^2\theta-a^3}{c^2+ca\theta+2ca+a^2\theta+2a^2} = c - \frac{c+a}{\theta+(c+a)/a+a/(c+a)} \geq c - \frac{c+a}{\theta+2}.$$

Adding the three inequalities above, we get

$$\begin{aligned}
&\frac{a^3+a^2b\theta+a^2b+ab^2\theta-b^3}{a^2+ab\theta+2ab+b^2\theta+2b^2}+\frac{b^3+b^2c\theta+b^2c+bc^2\theta-c^3}{b^2+bc\theta+2bc+c^2\theta+2c^2}\\
&+\frac{c^3+c^2a\theta+c^2a+ca^2\theta-a^3}{c^2+ca\theta+2ca+a^2\theta+2a^2}\\
&\geq a+b+c-2\cdot\frac{a+b+c}{\theta+2}=\frac{\theta}{\theta+2}(a+b+c).
\end{aligned}$$

This ends the proof. □

Some special cases of Theorem 2.2 are presented below.

If we take $\theta = 0$, then we have

$$\frac{a^3+a^2b-b^3}{a^2+2ab+2b^2}+\frac{b^3+b^2c-c^3}{b^2+2bc+2c^2}+\frac{c^3+c^2a-a^3}{c^2+2ca+2a^2}\geq 0,$$

which is not trivial at first glance.

If we take $\theta = 1$, then we have

$$\begin{aligned}
&\frac{a^3+2a^2b+ab^2-b^3}{a^2+3ab+3b^2}+\frac{b^3+2b^2c+bc^2-c^3}{b^2+3bc+3c^2}+\frac{c^3+2c^2a+ca^2-a^3}{c^2+3ca+3a^2}\\
&\geq\frac{1}{3}(a+b+c).
\end{aligned}$$

Naturally, other integer or decimal values of θ can be handled in a similar fashion.

Our third one-parameter family of cyclic inequalities is stated in the theorem below.

Theorem 2.3. *For any $a, b, c > 0$ and $\theta > -2$, we have*

$$\begin{aligned}
&\frac{a^2(a\theta+a+b\theta)}{a^2\theta+2a^2+ab\theta+2ab+b^2}+\frac{b^2(b\theta+b+c\theta)}{b^2\theta+2b^2+bc\theta+2bc+c^2}\\
&+\frac{c^2(c\theta+c+a\theta)}{c^2\theta+2c^2+ca\theta+2ca+a^2}\geq\frac{\theta}{\theta+2}(a+b+c).
\end{aligned}$$

Proof. Using a non-trivial decomposition and the inequality $x + 1/x \geq 2$ for any $x > 0$, we have

$$\frac{a^2(a\theta+a+b\theta)}{a^2\theta+2a^2+ab\theta+2ab+b^2}=a-\frac{a+b}{\theta+(a+b)/a+a/(a+b)}\geq a-\frac{a+b}{\theta+2}.$$

Similarly, we have

$$\frac{b^2(b\theta+b+c\theta)}{b^2\theta+2b^2+bc\theta+2bc+c^2}=b-\frac{b+c}{\theta+(b+c)/b+b/(b+c)}\geq b-\frac{b+c}{\theta+2}$$

and

$$\frac{c^2(c\theta+c+a\theta)}{c^2\theta+2c^2+ca\theta+2ca+a^2}=c-\frac{c+a}{\theta+(c+a)/c+c/(c+a)}\geq c-\frac{c+a}{\theta+2}.$$

Adding the three inequalities above, we get

$$\begin{aligned}&\frac{a^2(a\theta+a+b\theta)}{a^2\theta+2a^2+ab\theta+2ab+b^2}+\frac{b^2(b\theta+b+c\theta)}{b^2\theta+2b^2+bc\theta+2bc+c^2}\\&+\frac{c^2(c\theta+c+a\theta)}{c^2\theta+2c^2+ca\theta+2ca+a^2}\\&\geq a+b+c-2\cdot\frac{a+b+c}{\theta+2}=\frac{\theta}{\theta+2}(a+b+c).\end{aligned}$$

This completes the proof. □

In particular, if we take $\theta=1$, then we have

$$\frac{a^2(2a+b)}{3a^2+3ab+b^2}+\frac{b^2(2b+c)}{3b^2+3bc+c^2}+\frac{c^2(2c+a)}{3c^2+3ca+a^2}\geq\frac{1}{3}(a+b+c).$$

Naturally, other integer or decimal values of θ can be handled in a similar fashion.

We conclude this paper with a family of cyclic inequalities that we leave as a conjecture. For any $a,b,c>0$ and $\theta>-2$, we have

$$\begin{aligned}&\frac{a^3+a^2b\theta+ab^2\theta+ab^2+b^3}{a^2+ab\theta+b^2}+\frac{b^3+b^2c\theta+bc^2\theta+bc^2+c^3}{b^2+bc\theta+c^2}\\&+\frac{c^3+c^2a\theta+ca^2\theta+ca^2+a^3}{c^2+ca\theta+a^2}\geq\frac{2\theta+3}{\theta+2}(a+b+c).\end{aligned}$$

ACKNOWLEDGMENT

The author would like to thank the reviewers for their constructive comments.

REFERENCES

[1] M. BENCZE, C. ZHAO: *A refinement of Nesbitt's inequality*, Octogon Math. Mag., **16**(1A) (2008), 275–276.

[2] M. BENCZE, O. T. POP: *Generalizations and refinements for Nesbitt's inequality*, J. Math. Inequal., **5**(1) (2011), 13–20.

[3] V. CÎRTOAJE: *Mathematical Inequalities – Volume 3: Cyclic and Noncyclic Inequalities*, Lambert Academic Publishing, 2018.

[4] M. O. DRÂMBE: *Inequalities – Ideas and Methods*, Ed. Gil, Zalău, 2003.

[5] A. M. NESBITT: *Problem 15114*, Educational Times, **3** (1903), 37–38.

[6] F. H. WEI, S. H. WU: *Generalizations and analogues of Nesbitt's inequality*, Octogon Math. Mag., **17**(1) (2009), 215–220.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CAEN-NORMANDIE
UFR DES SCIENCES - CAMPUS 2, CAEN
FRANCE.
Email address: christophe.chesneau@gmail.com