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AN ALTERNATIVE PROOF FOR AN EXTENSION OF THE NESBITT INEQUALITY

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ABSTRACT. This paper investigates a parametric generalization of the Nesbitt inequality and provides an alternative proof based on the Jensen inequality. A new cyclic product inequality is also derived.

1. INTRODUCTION

One of the most classical and well-known cyclic inequalities is the Nesbitt inequality. It states that, for any $a, b, c > 0$, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

A parametric generalization of this inequality was established in [6, Theorem 1] by introducing an adjustable parameter k. Specifically, for any $a, b, c, k > 0$, we have

$$\frac{a}{kb+c} + \frac{b}{kc+a} + \frac{c}{ka+b} \geq \frac{3}{k+1}.$$

The proof is based on the Cauchy-Schwarz inequality. Another parametric generalization was established in [7, Corollary 8] by introducing an adjustable parameter λ. Specifically, for any $a, b, c > 0$ and $\lambda \in [0, 1]$, we have

$$\frac{a}{b+c+\lambda a} + \frac{b}{c+a+\lambda b} + \frac{c}{a+b+\lambda c} \geq \frac{3}{2+\lambda}.$$

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The proof is also based on the Cauchy-Schwarz inequality. Further developments and recent advances concerning the Nesbitt inequality can be found in [1–7]. In this paper, we contribute to [7, Corollary 8] by providing an alternative proof based on the Jensen inequality. A new cyclic product inequality is also derived.

2. Result

We formulate [7, Corollary 8], followed by the new proof.

Theorem 2.1. *[7, Corollary 8] Let $a, b, c > 0$ and $\lambda \in [0, 1]$. Then, we have*
$$\frac{a}{b+c+\lambda a} + \frac{b}{c+a+\lambda b} + \frac{c}{a+b+\lambda c} \geq \frac{3}{2+\lambda}.$$
Equality holds if and only if $a = b = c$.

New proof. Let $s = a + b + c$. We define
$$x_a = \frac{a}{s}, \quad x_b = \frac{b}{s}, \quad x_c = \frac{c}{s}.$$
Then, we have
$$\frac{a}{b+c+\lambda a} = \frac{a}{b+c+a-(1-\lambda)a} = \frac{x_a}{1-(1-\lambda)x_a}.$$
Set $k = 1 - \lambda$. We define
$$F(x) = \frac{x}{1-kx}.$$
Using standard differentiation rules and $k \in [0, 1]$, we compute
$$F''(x) = \frac{2k}{(1-kx)^3} \geq 0.$$
So F is convex on $(0, 1)$.

By the Jensen inequality and $\sum_{i\in\{a,b,c\}} x_i = s/s = 1$, we have
$$\frac{1}{3}\sum_{i\in\{a,b,c\}} F(x_i) \geq F\left(\frac{1}{3}\sum_{i\in\{a,b,c\}} x_i\right) = F\left(\frac{1}{3}\right).$$
Thus, we obtain
$$\sum_{i\in\{a,b,c\}} \frac{x_i}{1-kx_i} \geq 3\cdot\frac{1/3}{1-k/3} = \frac{1}{1-k/3}.$$

Substituting $k = 1 - \lambda$ gives

$$1 - \frac{k}{3} = 1 - \frac{1-\lambda}{3} = \frac{2+\lambda}{3}.$$

Hence, we have

$$\sum_{i \in \{a,b,c\}} \frac{x_i}{1-(1-\lambda)x_i} \geq \frac{3}{2+\lambda},$$

so that

$$\frac{a}{b+c+\lambda a} + \frac{b}{c+a+\lambda b} + \frac{c}{a+b+\lambda c} \geq \frac{3}{2+\lambda}.$$

Equality holds when $x_a = x_b = x_c = 1/3$, i.e., $a = b = c$. This completes the proof. □

Theorem 2.1 thus contributes to the literature by demonstrating how the Jensen inequality can be used to establish Nesbitt-type inequalities.

As a consequence of [7, Corollary 8], a new cyclic product inequality is derived below.

Theorem 2.2. *Let* $a, b, c > 0$. *Then, we have*

$$\left(1 + \frac{a}{b+c}\right)\left(1 + \frac{b}{c+a}\right)\left(1 + \frac{c}{a+b}\right) \geq \left(\frac{3}{2}\right)^3.$$

Proof. It follows from Theorem 2.1 that, for any $\lambda \in [0,1]$,

$$\frac{a}{b+c+\lambda a} + \frac{b}{c+a+\lambda b} + \frac{c}{a+b+\lambda c} \geq \frac{3}{2+\lambda}.$$

Integrating both sides with respect to λ with $\lambda \in [0,1]$ yields

$$\int_0^1 \left(\frac{a}{b+c+\lambda a} + \frac{b}{c+a+\lambda b} + \frac{c}{a+b+\lambda c}\right) d\lambda \geq \int_0^1 \frac{3}{2+\lambda} d\lambda,$$

so that

$$\int_0^1 \frac{a}{b+c+\lambda a} d\lambda + \int_0^1 \frac{b}{c+a+\lambda b} d\lambda + \int_0^1 \frac{c}{a+b+\lambda c} d\lambda \geq \int_0^1 \frac{3}{2+\lambda} d\lambda,$$

and

$$[\ln(b+c+\lambda a)]_{\lambda=0}^{\lambda=1} + [\ln(c+a+\lambda b)]_{\lambda=0}^{\lambda=1} + [\ln(a+b+\lambda c)]_{\lambda=0}^{\lambda=1} \geq 3\,[\ln(2+\lambda)]_{\lambda=0}^{\lambda=1},$$

which gives

$$\ln\left(1+\frac{a}{b+c}\right)+\ln\left(1+\frac{b}{c+a}\right)+\ln\left(1+\frac{c}{a+b}\right)\geq 3\ln\left(\frac{3}{2}\right).$$

Using standard manipulations of the logarithmic function, we get

$$\ln\left(\left(1+\frac{a}{b+c}\right)\left(1+\frac{b}{c+a}\right)\left(1+\frac{c}{a+b}\right)\right)\geq \ln\left(\left(\frac{3}{2}\right)^3\right),$$

so that

$$\left(1+\frac{a}{b+c}\right)\left(1+\frac{b}{c+a}\right)\left(1+\frac{c}{a+b}\right)\geq \left(\frac{3}{2}\right)^3.$$

This concludes the proof. □

This result may be viewed as a product-type variant of the Nesbitt inequality.

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