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# THREE ONE-PARAMETER FAMILIES OF CYCLIC INEQUALITIES PROVED BY THE TITU LEMMA

Christophe Chesneau

ABSTRACT. In this paper, we investigate three one-parameter families of cyclic inequalities. The originality of our proofs lies in their reliance on the Titu lemma.

## 1. INTRODUCTION

Cyclic inequalities arise frequently in the investigation of fractional inequalities and often serve as useful tools in their proofs. Among the most celebrated examples is the Nesbitt inequality, which we recall below. For any $a, b, c > 0$, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2},$$

sometimes written with the sigma cyclic sum as follows:

$$\sum_{cyc} \frac{a}{b+c} \geq \frac{3}{2}.$$

This inequality was generalized in a one-parameter family in [7], which we recall below. For any $a, b, c, k > 0$, we have

$$\frac{a}{kb+c} + \frac{b}{kc+a} + \frac{c}{ka+b} \geq \frac{3}{k+1}.$$

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Numerous proofs, extensions, refinements, and related results concerning this inequality have appeared in the literature; see, for example, [1–5, 7].

In this paper, following the parametric approach developed in [7], we establish three one-parameter families of cyclic inequalities with lower bounds of the form

$$\frac{a^2+b^2+c^2}{\ell},$$

where $\ell$ denotes a certain constant. A distinctive feature of our proofs is their systematic use of the Titu lemma. In a sense, these results complete the collection of cyclic inequalities presented in [3].

The three one-parameter family of cyclic inequalities are developed in Section 2. A supplementary result is given in Section 3.

## 2. Cyclic inequalities

By a direct application of the Titu lemma, one can prove that, for any $a, b, c, k > 0$, the following cyclic inequality holds:

$$\frac{a^2}{b+kc}+\frac{b^2}{c+ka}+\frac{c^2}{a+kb} \geq \frac{a+b+c}{k+1}.$$

Inspired by this result, the theorem below presents our first one-parameter family of cyclic inequalities.

**Theorem 2.1.** *For any $a, b, c, k > 0$, the following cyclic inequality holds:*

$$\frac{a^3}{b+kc}+\frac{b^3}{c+ka}+\frac{c^3}{a+kb} \geq \frac{a^2+b^2+c^2}{k+1}.$$

*Proof.* First of all, the Titu lemma states that, for any $x_1, x_2, x_3 \in \mathbb{R}$ and $y_1, y_2, y_3 > 0$, we have

$$\frac{x_1^2}{y_1}+\frac{x_2^2}{y_2}+\frac{x_3^2}{y_3} \geq \frac{(x_1+x_2+x_3)^2}{y_1+y_2+y_3}. \tag{2.1}$$

See [6].

Slightly modifying the main term, applying a suitably chosen form of the Titu lemma and simplifying the denominator, we get

$$\frac{a^3}{b+kc}+\frac{b^3}{c+ka}+\frac{c^3}{a+kb} = \frac{a^4}{ab+kac}+\frac{b^4}{bc+kba}+\frac{c^4}{ca+kcb}$$

$$\geq \frac{(a^2+b^2+c^2)^2}{(ab+kac)+(bc+kba)+(ca+kcb)}$$

$$= \frac{(a^2+b^2+c^2)^2}{(k+1)(ab+bc+ca)}. \tag{2.2}$$

Using the elementary inequality

$$ab+bc+ca \leq \left(\frac{a^2+b^2}{2}\right)+\left(\frac{b^2+c^2}{2}\right)+\left(\frac{c^2+a^2}{2}\right)=a^2+b^2+c^2, \tag{2.3}$$

we get

$$\frac{(a^2+b^2+c^2)^2}{(k+1)(ab+bc+ca)} \geq \frac{(a^2+b^2+c^2)^2}{(k+1)(a^2+b^2+c^2)} = \frac{a^2+b^2+c^2}{k+1}. \tag{2.4}$$

Putting Equations (2.2) and (2.4) together, we obtain

$$\frac{a^3}{b+kc}+\frac{b^3}{c+ka}+\frac{c^3}{a+kb} \geq \frac{a^2+b^2+c^2}{k+1}.$$

This concludes the proof. $\square$

Another formulation of the result is

$$\sum_{cyc} \frac{a^3}{b+kc} \geq \frac{1}{k+1}\sum_{cyc} a^2.$$

In particular, if we take $k=1$, then we have

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b} \geq \frac{a^2+b^2+c^2}{2}.$$

If we take $k=2$, then we have

$$\frac{a^3}{b+2c}+\frac{b^3}{c+2a}+\frac{c^3}{a+2b} \geq \frac{a^2+b^2+c^2}{3}.$$

The theorem below presents our second one-parameter family of cyclic inequalities.

**Theorem 2.2.** *For any $a, b, c, k > 0$, the following cyclic inequality holds:*

$$\frac{a^4}{a^2+kab+b^2}+\frac{b^4}{b^2+kbc+c^2}+\frac{c^4}{c^2+kca+a^2} \geq \frac{a^2+b^2+c^2}{k+2}.$$

*Proof.* Applying a suitably chosen form of the Titu lemma in Equation (2.1) and simplifying the denominator, we get

$$\begin{aligned} &\frac{a^4}{a^2+kab+b^2}+\frac{b^4}{b^2+kbc+c^2}+\frac{c^4}{c^2+kca+a^2} \\ &\geq \frac{(a^2+b^2+c^2)^2}{(a^2+kab+b^2)+(b^2+kbc+c^2)+(c^2+kca+a^2)} \\ &= \frac{(a^2+b^2+c^2)^2}{2(a^2+b^2+c^2)+k(ab+bc+ca)}. \end{aligned} \tag{2.5}$$

Using the inequality in Equation (2.3), we have

$$\begin{aligned} &\frac{(a^2+b^2+c^2)^2}{2(a^2+b^2+c^2)+k(ab+bc+ca)} \\ &\geq \frac{(a^2+b^2+c^2)^2}{2(a^2+b^2+c^2)+k(a^2+b^2+c^2)} \\ &= \frac{a^2+b^2+c^2}{k+2}. \end{aligned} \tag{2.6}$$

Putting Equations (2.5) and (2.6) together, we obtain

$$\frac{a^4}{a^2+kab+b^2}+\frac{b^4}{b^2+kbc+c^2}+\frac{c^4}{c^2+kca+a^2} \geq \frac{a^2+b^2+c^2}{k+2}.$$

This concludes the proof. $\square$

Another formulation of the result is

$$\sum_{cyc}\frac{a^4}{a^2+kab+b^2} \geq \frac{1}{k+2}\sum_{cyc}a^2.$$

In particular, if we take $k=1$, then we have

$$\frac{a^4}{a^2+ab+b^2}+\frac{b^4}{b^2+bc+c^2}+\frac{c^4}{c^2+ca+a^2} \geq \frac{a^2+b^2+c^2}{3}.$$

If we take $k=2$, then we have

$$\frac{a^4}{a^2+2ab+b^2}+\frac{b^4}{b^2+2bc+c^2}+\frac{c^4}{c^2+2ca+a^2} \geq \frac{a^2+b^2+c^2}{4}.$$

The theorem below presents our third one-parameter family of cyclic inequalities.

**Theorem 2.3.** *For any $a, b, c, k > 0$, the following cyclic inequality holds:*

$$\frac{a^4}{a^2 + k(ab + bc)} + \frac{b^4}{b^2 + k(bc + ca)} + \frac{c^4}{c^2 + k(ca + ab)} \geq \frac{a^2 + b^2 + c^2}{2k + 1}.$$

*Proof.* Applying a suitably chosen form of the Titu lemma in Equation (2.1) and simplifying the denominator, we get

$$\begin{aligned} &\frac{a^4}{a^2 + k(ab + bc)} + \frac{b^4}{b^2 + k(bc + ca)} + \frac{c^4}{c^2 + k(ca + ab)} \\ &\geq \frac{(a^2 + b^2 + c^2)^2}{(a^2 + k(ab + bc)) + (b^2 + k(bc + ca)) + (c^2 + k(ca + ab))} \\ &= \frac{(a^2 + b^2 + c^2)^2}{(a^2 + b^2 + c^2) + 2k(ab + bc + ca)}. \end{aligned} \tag{2.7}$$

Using the inequality in Equation (2.3), we have

$$\begin{aligned} &\frac{(a^2 + b^2 + c^2)^2}{(a^2 + b^2 + c^2) + 2k(ab + bc + ca)} \\ &\geq \frac{(a^2 + b^2 + c^2)^2}{(a^2 + b^2 + c^2) + 2k(a^2 + b^2 + c^2)} \\ &= \frac{a^2 + b^2 + c^2}{2k + 1}. \end{aligned} \tag{2.8}$$

Putting Equations (2.7) and (2.8) together, we obtain

$$\frac{a^4}{a^2 + k(ab + bc)} + \frac{b^4}{b^2 + k(bc + ca)} + \frac{c^4}{c^2 + k(ca + ab)} \geq \frac{a^2 + b^2 + c^2}{2k + 1}.$$

This concludes the proof. □

Another formulation of the result is

$$\sum_{cyc} \frac{a^4}{a^2 + k(ab + bc)} \geq \frac{1}{2k + 1} \sum_{cyc} a^2.$$

In particular, if we take $k = 1$, then we have

$$\frac{a^4}{a^2 + ab + bc} + \frac{b^4}{b^2 + bc + ca} + \frac{c^4}{c^2 + ca + ab} \geq \frac{a^2 + b^2 + c^2}{3}.$$

If we take $k = 2$, then we have

$$\frac{a^4}{a^2 + 2(ab + bc)} + \frac{b^4}{b^2 + 2(bc + ca)} + \frac{c^4}{c^2 + 2(ca + ab)} \geq \frac{a^2 + b^2 + c^2}{5}.$$

## 3. A SUPPLEMENTARY RESULT

During our study of cyclic inequalities, we encountered another new elegant inequality. Although it differs from the preceding ones, it relies on the same underlying proof techniques. It is presented below.

**Theorem 3.1.** *For any* $a, b, c > 0$*, the following cyclic inequality holds:*

$$\frac{a^2}{ab+3bc+2ca}+\frac{b^2}{bc+3ca+2ab}+\frac{c^2}{ca+3ab+2bc}\geq\frac{1}{2}.$$

*Proof.* Applying a suitably chosen form of the Titu lemma in Equation (2.1) and simplifying the denominator, we get

$$\begin{aligned}&\frac{a^2}{ab+3bc+2ca}+\frac{b^2}{bc+3ca+2ab}+\frac{c^2}{ca+3ab+2bc}\\ &\geq\frac{(a+b+c)^2}{(ab+3bc+2ca)+(bc+3ca+2ab)+(ca+3ab+2bc)}\\ &=\frac{(a+b+c)^2}{6(ab+bc+ca)}.\end{aligned}\tag{3.1}$$

Using the inequality in Equation (2.3), we have

$$\begin{aligned}\frac{(a+b+c)^2}{6(ab+bc+ca)}&=\frac{(a^2+b^2+c^2)+2(ab+bc+ca)}{6(ab+bc+ca)}\\ &=\frac{a^2+b^2+c^2}{6(ab+bc+ca)}+\frac{1}{3}\geq\frac{1}{6}+\frac{1}{3}=\frac{1}{2}.\end{aligned}\tag{3.2}$$

Putting Equations (3.1) and (3.2) together, we obtain

$$\frac{a^2}{ab+3bc+2ca}+\frac{b^2}{bc+3ca+2ab}+\frac{c^2}{ca+3ab+2bc}\geq\frac{1}{2}.$$

This concludes the proof. □

Another formulation of the result is

$$\sum_{cyc}\frac{a^2}{ab+3bc+2ca}\geq\frac{1}{2}.$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CAEN-NORMANDIE
UFR DES SCIENCES - CAMPUS 2, CAEN
FRANCE.
*Email address*: christophe.chesneau@gmail.com