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ON THE INTEGRAL VERSIONS OF THE BERGSTRÖM AND MILNE INEQUALITIES

Christophe Chesneau

ABSTRACT. In this paper, we investigate the integral versions of the Bergström and Milne inequalities.

1. INTRODUCTION

Among the classical inequalities involving weighted sums and sums of coefficients, two of the most fundamental are the Bergström and Milne inequalities. These inequalities have played an important role in the development of the theory of algebraic inequalities and have found numerous applications in analysis, optimization, and related areas.

- The Bergström inequality is stated as follows. Let $n \in \mathbb{N} \backslash \{0\}$, $x_1, \ldots, x_n \in \mathbb{R}$ and $a_1, \ldots, a_n > 0$. Then, we have

$$\frac{\left(\sum_{i=1}^{n} x_i\right)^2}{\sum_{i=1}^{n} a_i} \leq \sum_{i=1}^{n} \frac{x_i^2}{a_i}.$$

  See [1].

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- The Milne inequality is stated as follows. Let $n \in \mathbb{N}\backslash\{0\}$, $x_1, \ldots, x_n > 0$ and $y_1, \ldots, y_n > 0$. Then, we have

$$\sum_{i=1}^{n} \frac{x_i y_i}{x_i + y_i} \leq \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i}.$$

The main objective of this paper is to derive integral analogues of the Bergström and Milne inequalities. To our knowledge, these findings have not been previously reported.

2. Results with proofs

A Bergström-type integral inequality is given below.

Theorem 2.1. *Let $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $b > a$, and $f : (a, b) \to \mathbb{R}$, $g : (a, b) \to [0, +\infty)$ be two integrable functions such that $\int_a^b g(x)dx > 0$. Then, the following inequality holds:*

$$\frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b g(x)dx} \leq \int_a^b \frac{f^2(x)}{g(x)} dx.$$

Proof. By an appropriate development and the use of the Cauchy-Schwarz integral inequality, we get

$$\left(\int_a^b f(x)dx\right)^2 = \left(\int_a^b \frac{f(x)}{\sqrt{g(x)}} \sqrt{g(x)} dx\right)^2$$
$$\leq \left(\int_a^b \frac{f^2(x)}{g(x)} dx\right)\left(\int_a^b g(x)dx\right).$$

Since $\int_a^b g(x)dx > 0$, this can be expressed as follows:

$$\frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b g(x)dx} \leq \int_a^b \frac{f^2(x)}{g(x)} dx.$$

This concludes the proof. □

A Milne-type integral inequality, which is the main contribution of the paper, is given below. The proof is based on the Bergström-type integral inequality in Theorem 2.1.

Theorem 2.2. *Let $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $b > a$, and $f, g : (a, b) \to [0, +\infty)$ be two integrable functions such that $\int_a^b f(x)dx + \int_a^b g(x)dx > 0$. Then, the following inequality holds:*

$$\int_a^b \frac{f(x)g(x)}{f(x)+g(x)}dx \leq \frac{\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)}{\int_a^b f(x)dx + \int_a^b g(x)dx}.$$

Proof. Using appropriate fractional and integral developments, and the Bergström-type integral inequality in Theorem 2.1, we find that

$$\begin{aligned}
&\int_a^b \frac{f(x)g(x)}{f(x)+g(x)}dx = \int_a^b \frac{f(x)(f(x)+g(x)) - f^2(x)}{f(x)+g(x)}dx \\
&= \int_a^b \left(f(x) - \frac{f^2(x)}{f(x)+g(x)}\right) dx \\
&= \int_a^b f(x)dx - \int_a^b \frac{f^2(x)}{f(x)+g(x)}dx \\
&\overset{Theorem\ 2.1}{\leq} \int_a^b f(x)dx - \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b (f(x)+g(x))dx} \\
&= \int_a^b f(x)dx - \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b f(x)dx + \int_a^b g(x)dx} \\
&= \frac{\left(\int_a^b f(x)dx\right)\left(\int_a^b f(x)dx + \int_a^b g(x)dx\right) - \left(\int_a^b f(x)dx\right)^2}{\int_a^b f(x)dx + \int_a^b g(x)dx} \\
&= \frac{\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)}{\int_a^b f(x)dx + \int_a^b g(x)dx}.
\end{aligned}$$

This concludes the proof. □

The structural distribution of the integral varies noticeably between the upper and lower bounds.

As a direction for future research, it would be of interest to extend the Milne integral inequality to cases involving three functions. By introducing a third function h, we propose the following logical conjecture:

$$\int_a^b \frac{f(x)g(x)h(x)}{f(x)+g(x)+h(x)}dx \leq \frac{\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)\left(\int_a^b h(x)dx\right)}{\int_a^b f(x)dx + \int_a^b g(x)dx + \int_a^b h(x)dx}.$$

As a final remark, Theorems 2.1 and 2.2 can be extended to multidimensional functions by a straightforward argument.

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Department of Mathematics
University of Caen-Normandie
UFR des Sciences - Campus 2, Caen
France.
Email address: christophe.chesneau@gmail.com