J COMP SCI APPL MATH
Journal of Computer Science and Applied Mathematics
Vol. 8, no.2, (2026), 229–237
ISSN: 1857-9582 https://doi.org/10.37418/jcsam.8.2.7

THE NESBITT INEQUALITY: AN INTEGRAL APPROACH

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ABSTRACT. This paper proposes an integral approach to one of the most celebrated inequalities in mathematics, namely the Nesbitt inequality.

1. INTRODUCTION

The Nesbitt inequality is a famous inequality in mathematics. A formal statement of this inequality is given in the theorem below.

Theorem 1.1 (The Nesbitt inequality)**.** *For any* $a, b, c > 0$*, we have*

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$

It is a classical result in the theory of inequalities and serves as an application of techniques, such as the Cauchy-Schwarz inequality, the Jensen inequality, the Arithmetic-Geometric Mean (AM-GM) inequality, and substitution methods. The inequality is widely studied in mathematical olympiads and undergraduate mathematics because of its simplicity in statement and the variety of proofs it admits. For further developments and related results, we refer the reader to [1, 2, 4, 5].

In this paper, we revisit the Nesbitt inequality from the perspective of integral representations, an approach that appears to have received little attention

Key words and phrases. Nesbitt inequality, integrals.
Submitted: 30.06.2026; *Accepted:* 14.07.2026; *Published:* 18.07.2026.

in the literature. This viewpoint leads naturally to a new one-parameter generalization of the Nesbitt inequality, which constitutes the main contribution of the paper (Section 2). In addition, for pedagogical purposes, we extract from our approach a concise, half-page proof of the classical Nesbitt inequality based solely on elementary properties of integrals and the Jensen inequality (Section 3). The paper concludes with a generalization of our main result to the case of more than three variables (Section 4). The appendix provides an alternative perspective based on convexity and the Jensen inequality.

2. A GENERALIZATION OF THE NESBITT INEQUALITY

Our generalization of the Nesbitt inequality introduces an additional parameter p, as stated in the following theorem.

Theorem 2.1. *For any $a, b, c, p > 0$, the following inequality holds:*

$$\frac{a}{(b+c)^p} + \frac{b}{(a+c)^p} + \frac{c}{(a+b)^p} \geq \left(\frac{3}{2}\right)^p \frac{1}{(a+b+c)^{p-1}}.$$

Note: The Nesbitt inequality emerges by taking $p = 1$.

Proof. Given the novelty of the approach, the proof is presented step by step. We begin with the standard integral definition of the gamma function, $\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt$, which allows us to express inverse powers as an exponential integral. For any $X > 0$, we have

$$\frac{1}{X^p} = \frac{1}{\Gamma(p)} \int_0^\infty t^{p-1}e^{-Xt}dt.$$

We apply this identity to the three terms in our inequality, setting X to $b + c$, $a + c$, and $a + b$, respectively. Multiplying by their respective numerators a, b, and c, we get

$$\frac{a}{(b+c)^p} = \frac{1}{\Gamma(p)} \int_0^\infty at^{p-1}e^{-(b+c)t}dt,$$
$$\frac{b}{(a+c)^p} = \frac{1}{\Gamma(p)} \int_0^\infty bt^{p-1}e^{-(a+c)t}dt$$

and

$$\frac{c}{(a+b)^p} = \frac{1}{\Gamma(p)} \int_0^\infty ct^{p-1}e^{-(a+b)t}dt.$$

Summing these three terms gives us an exact integral representation for the left-hand side (LHS) of our inequality, that is

$$\begin{aligned}\text{LHS} &= \frac{a}{(b+c)^p} + \frac{b}{(a+c)^p} + \frac{c}{(a+b)^p}\\ &= \frac{1}{\Gamma(p)} \int_0^\infty t^{p-2} \left[ate^{-(b+c)t} + bte^{-(a+c)t} + cte^{-(a+b)t}\right] dt.\end{aligned}$$

To analyze the expression inside the brackets, let us define new variables based on scaled versions of $a, b,$ and c. Let $x = at$, $y = bt$, and $z = ct$. The bracketed expression becomes a function $F(x, y, z)$, as follows:

$$F(x, y, z) = xe^{-(y+z)} + ye^{-(x+z)} + ze^{-(x+y)}.$$

Notice that we can multiply and divide each term in $F(x, y, z)$ by e^x, e^y, and e^z, respectively to factor out a common exponential. Let $S = x + y + z$. We have

$$F(x, y, z) = e^{-(x+y+z)} \left(xe^x + ye^y + ze^z\right) = e^{-S} \left(xe^x + ye^y + ze^z\right).$$

Now, consider the function $g(u) = ue^u$ for $u > 0$. Taking the second derivative yields

$$g'(u) = e^u(u + 1)$$

and

$$g''(u) = e^u(u + 2).$$

Since $u > 0$, we have $g''(u) > 0$, proving that $g(u)$ is strictly convex. Therefore, we can apply the Jensen inequality to the sum $xe^x + ye^y + ze^z$. We have

$$\begin{aligned}\frac{xe^x + ye^y + ze^z}{3} &= \frac{g(x) + g(y) + g(z)}{3} \geq g\left(\frac{x + y + z}{3}\right)\\ &= \left(\frac{x + y + z}{3}\right) e^{(x+y+z)/3},\end{aligned}$$

so that

$$xe^x + ye^y + ze^z \geq Se^{S/3}.$$

Substituting this lower bound back into $F(x, y, z)$, we get

$$F(x, y, z) \geq e^{-S} \left(Se^{S/3}\right) = Se^{-(2/3)S}.$$

Converting our scaled variables back to a, b, c, and t (where $S = (a+b+c)t$), we obtain

$$F(at, bt, ct) \geq (a+b+c)te^{-(2/3)(a+b+c)t}.$$

Now, substituting this strictly pointwise lower bound back into our main integral, we get

$$\text{LHS} \geq \frac{1}{\Gamma(p)} \int_0^\infty t^{p-2} \left[(a+b+c)te^{-(2/3)(a+b+c)t}\right] dt.$$

Simplifying the integral by grouping the t terms, we obtain

$$\text{LHS} \geq \frac{a+b+c}{\Gamma(p)} \int_0^\infty t^{p-1} e^{-(2/3)(a+b+c)t} dt.$$

This integral is simply another gamma function integral of the form

$$\int_0^\infty t^{p-1} e^{-\alpha t} dt = \frac{\Gamma(p)}{\alpha^p},$$

where the decay constant is $\alpha = (2/3)(a+b+c)$. Evaluating the integral yields

$$\text{LHS} \geq \frac{a+b+c}{\Gamma(p)} \left[\frac{\Gamma(p)}{((2/3)(a+b+c))^p}\right].$$

The $\Gamma(p)$ terms cancel out. Rearranging the remaining algebraic terms gives us the final bound, as follows:

$$\text{LHS} \geq \frac{a+b+c}{((2/3))^p (a+b+c)^p} = \left(\frac{3}{2}\right)^p \frac{1}{(a+b+c)^{p-1}}.$$

This concludes the proof. □

To the best of the knowledge of the author, both the result stated in Theorem 2.1 and the proof technique developed herein are new to the literature.

Notably, the constant factor of $3/2$ emerges naturally throughout the proof, which requires only the Jensen inequality for a basic convex function as an intermediate tool.

As illustrative special cases, observe that, for $p = 1/2$, Theorem 2.1 yields

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{a+c}} + \frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}}\sqrt{a+b+c}$$

for $p = 2$, it gives

$$\frac{a}{(b+c)^2} + \frac{b}{(a+c)^2} + \frac{c}{(a+b)^2} \geq \left(\frac{3}{2}\right)^2 \frac{1}{a+b+c}$$

and, for $p = 3$, it gives

$$\frac{a}{(b+c)^3} + \frac{b}{(a+c)^3} + \frac{c}{(a+b)^3} \geq \left(\frac{3}{2}\right)^3 \frac{1}{(a+b+c)^2}.$$

3. A half-page integral proof of the Nesbitt inequality

For pedagogical purposes, we present a concise half-page proof of the Nesbitt inequality based on elementary integral arguments and the Jensen inequality.

Proof of Theorem 1.1 with the integral approach. We have

$$\begin{aligned}
&\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \\
&= a\int_0^{\infty} e^{-(b+c)t}dt + b\int_0^{\infty} e^{-(a+c)t}dt + c\int_0^{\infty} e^{-(a+b)t}dt \\
&= \int_0^{\infty} \left[ae^{-(b+c)t} + be^{-(a+c)t} + ce^{-(a+b)t}\right] dt \\
&= \int_0^{\infty} \frac{1}{t}e^{-(a+b+c)t}(ate^{at} + bte^{bt} + cte^{ct})dt \\
&= \int_0^{\infty} \frac{1}{t}e^{-(a+b+c)t}3\left(\frac{ate^{at} + bte^{bt} + cte^{ct}}{3}\right) dt \\
&\overset{\text{Jensen inequality using } ue^u \text{ convex}}{\geq} \int_0^{\infty} \frac{1}{t}e^{-(a+b+c)t}3\left[\frac{at+bt+ct}{3}e^{(at+bt+ct)/3}\right] dt \\
&= (a+b+c)\int_0^{\infty} e^{-(2/3)(a+b+c)t}dt \\
&= (a+b+c)\left[-\frac{3}{2(a+b+c)}e^{-(2/3)(a+b+c)t}\right]_{t=0}^{t\to\infty} \\
&= (a+b+c)\frac{3}{2(a+b+c)} = \frac{3}{2}.
\end{aligned}$$

This ends the proof □

4. A generalization of Theorem 2.1

We offer a natural generalization of Theorem 2.1 with n variables, not only 3 variables, a, b, and c.

Theorem 4.1. *For any integer $n \geq 3$, $x_1, x_2, \ldots, x_n, p > 0$, and $S = \sum_{i=1}^{n} x_i$, the following inequality holds:*

$$\sum_{i=1}^{n} \frac{x_i}{(S - x_i)^p} \geq \left(\frac{n}{n-1} \right)^p \frac{1}{S^{p-1}}.$$

Note: Theorem 2.1 emerges by taking $n = 3$.

Proof. Using the techniques detailed in the proof of Theorem 2.1, we have

$$\sum_{i=1}^{n} \frac{x_i}{(S - x_i)^p} = \sum_{i=1}^{n} x_i \cdot \frac{1}{\Gamma(p)} \int_0^{\infty} t^{p-1} e^{-(S-x_i)t} dt$$
$$= \frac{1}{\Gamma(p)} \int_0^{\infty} t^{p-2} e^{-St} \left(\sum_{i=1}^{n} x_i t e^{x_i t} \right) dt$$
$$= \frac{1}{\Gamma(p)} \int_0^{\infty} t^{p-2} e^{-St} n \cdot \left(\frac{1}{n} \sum_{i=1}^{n} x_i t e^{x_i t} \right) dt$$
$$\overset{\textit{Jensen inequality using } ue^u \textit{ convex}}{\geq} \frac{1}{\Gamma(p)} \int_0^{\infty} t^{p-2} e^{-St} n \cdot \left[\left(\frac{1}{n} \sum_{i=1}^{n} x_i t \right) e^{(1/n) \sum_{i=1}^{n} x_i t} \right] dt$$
$$= S \cdot \frac{1}{\Gamma(p)} \int_0^{\infty} t^{p-1} e^{-St} e^{(1/n)St} dt$$
$$= S \cdot \frac{1}{\Gamma(p)} \int_0^{\infty} t^{p-1} e^{-[(n-1)/n]St} dt$$
$$= S \cdot \frac{1}{\Gamma(p)} \cdot \frac{\Gamma(p)}{[(n-1)/n]^p S^p} = \left(\frac{n}{n-1} \right)^p \frac{1}{S^{p-1}}.$$

This completes the proof. □

Besides yielding a short and transparent proof, the integral framework provides new insight into the structure of the inequality and suggests a broader family of related inequalities.

As a final remark, Theorem 4.1 admits a natural extension through the use of an increasing convex function f. Indeed, within the framework of the theorem, an immediate application of the Jensen inequality gives

$$\sum_{i=1}^{n} f\left(\frac{x_i}{(S-x_i)^p}\right) \geq nf\left(\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{(S-x_i)^p}\right)$$
$$\geq nf\left(\frac{1}{n}\left(\frac{n}{n-1}\right)^p \frac{1}{S^{p-1}}\right).$$

The freedom in the choice of f suggests several possible extensions and provides a promising avenue for deriving new families of cyclic inequalities.

Appendix

We conclude this paper with a general cyclic inequality derived from a problem solved in [3]. This finding generalizes several of our earlier results without relying on integration, making it of independent interest.

Theorem 4.2. *For any $a, b, c > 0$, and any convex and decreasing function $f : (0,\infty) \to \mathbb{R}$, the following inequality holds:*

$$af(b+c) + bf(a+c) + cf(a+b) \geq (a+b+c)f\left(\frac{2(a+b+c)}{3}\right).$$

Proof. Since f is convex, the Jensen inequality gives

$$\begin{aligned}
&af(b+c) + bf(a+c) + cf(a+b) \\
&= (a+b+c)\left(\frac{a}{a+b+c}f(b+c) + \frac{b}{a+b+c}f(a+c) + \frac{c}{a+b+c}f(a+b)\right) \\
&\geq (a+b+c)f\left(\frac{a(b+c)}{a+b+c} + \frac{b(a+c)}{a+b+c} + \frac{c(a+b)}{a+b+c}\right) \\
&= (a+b+c)f\left(\frac{a(b+c)+b(a+c)+c(a+b)}{a+b+c}\right) \\
&= (a+b+c)f\left(\frac{2(ab+bc+ac)}{a+b+c}\right).
\end{aligned}$$

Now, by the basic inequalities $a^2+b^2 \geq 2ab$, $b^2+c^2 \geq 2bc$ and $a^2+c^2 \geq 2ac$, we have

$$\begin{aligned}
&(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ac) \\
&= \left(\frac{a^2+b^2}{2}\right)+\left(\frac{b^2+c^2}{2}\right)+\left(\frac{a^2+c^2}{2}\right)+2(ab+bc+ac) \\
&\geq ab+bc+ac+2(ab+bc+ac) = 3(ab+bc+ac),
\end{aligned}$$

so that

$$ab+bc+ac \leq \frac{1}{3}(a+b+c)^2.$$

Using this and the fact that f is decreasing, we get

$$\begin{aligned}
&(a+b+c)f\left(\frac{2(ab+bc+ac)}{a+b+c}\right) \geq (a+b+c)f\left(\frac{2(a+b+c)^2}{3(a+b+c)}\right) \\
&= (a+b+c)f\left(\frac{2(a+b+c)}{3}\right).
\end{aligned}$$

Substituting this lower bound back, we get the desired result. This ends the proof. □

Note that

- if f is assumed to be concave and increasing instead of convex and decreasing in Theorem 4.2, the inequality is reversed,
- if we take $f(x) = 1/x$, which is convex and decreasing, then Theorem 4.2 gives the Nesbitt inequality,
- if we take $f(x) = 1/x^3$, which is convex and decreasing, then Theorem 4.2 gives the solution in [3]; Theorem 4.2 thus gives more general perspectives of applications,
- if we take $f(x) = 1/x^p$ with $p > 0$, which is convex and decreasing, then Theorem 4.2 yields Theorem 2.1.

Some other examples are developed below. We can consider $f(x) = e^{-x}$, which is convex and decreasing. Using Theorem 4.2, the following inequality holds:

$$ae^{-(b+c)} + be^{-(a+c)} + ce^{-(a+b)} \geq (a+b+c)e^{-2(a+b+c)/3}.$$

We can consider $f(x) = e^{1/x}$, which is convex and decreasing. Using Theorem 4.2, the following inequality holds:

$$ae^{1/(b+c)} + be^{1/(a+c)} + ce^{1/(a+b)} \geq (a+b+c)e^{3/(2(a+b+c))}.$$

We can consider $f(x) = \ln(1+e^{-x})$, which is convex and decreasing. Using Theorem 4.2, the following inequality holds:

$$\begin{aligned} &a\ln(1+e^{-(b+c)}) + b\ln(1+e^{-(a+c)}) + c\ln(1+e^{-(a+b)}) \\ &\geq (a+b+c)\ln(1+e^{-2(a+b+c)/3}), \end{aligned}$$

which is equivalent to

$$(1+e^{-(b+c)})^a(1+e^{-(a+c)})^b(1+e^{-(a+b)})^c \geq (1+e^{-2(a+b+c)/3})^{a+b+c}.$$

This cyclic product introduces a novel formula to the literature.

ACKNOWLEDGMENT

The author would like to thank the reviewers for their constructive comments.

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