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A COLLECTION OF SIX TWO-PARAMETER FAMILIES OF CYCLIC INEQUALITIES

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ABSTRACT. This paper investigates six two-parameter families of cyclic inequalities, establishing their validity through self-contained proofs and illustrative examples.

1. INTRODUCTION

Cyclic inequalities play a pivotal role in the study and verification of fractional algebraic inequalities. Among the most classical and celebrated benchmarks in this domain is the Nesbitt inequality, which states that, for any $a, b, c > 0$,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Over the decades, a vast body of literature has dedicated itself to discovering alternative proofs, extensions, refinements, and generalized counterparts to this fundamental result (see, for example, [1–7]).

In particular, in [3], three one-parameter families of cyclic inequalities are developed. As an example, we can consider [3, Theorem 1], which states that, for any $a, b, c > 0$ and $\theta > -2$,

$$\frac{a^2(a+b\theta-b)}{a^2+ab\theta+b^2} + \frac{b^2(b+c\theta-c)}{b^2+bc\theta+c^2} + \frac{c^2(c+a\theta-a)}{c^2+ca\theta+a^2} \geq \frac{\theta}{\theta+2}(a+b+c). \tag{1.1}$$

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Motivated by recent developments in the field, this paper introduces and establishes six new two-parameter families of cyclic inequalities. As we progress through these families, we explore structures of increasing algebraic complexity. Similar to Equation (1.1), the resulting lower bounds take the simple form $\gamma(a+b+c)$, although γ now depends on the two parameters involved. The next section presents our core results, complete with detailed proofs and illustrative examples.

2. Cyclic inequalities

Our first family of cyclic inequalities is presented in the theorem below. As in all theorems in this section, we emphasize the inclusion of the two parameters, k and ℓ.

Theorem 2.1. *For any* $a, b, c > 0$ *and* $k, \ell \geq 0$*, we have*

$$\frac{a\left(a^2+(1-\ell)b^2\right)}{a^2+kab+b^2}+\frac{b\left(b^2+(1-\ell)c^2\right)}{b^2+kbc+c^2}+\frac{c\left(c^2+(1-\ell)a^2\right)}{c^2+kca+a^2} \geq \frac{2-\ell}{2+k}(a+b+c).$$

Proof. Introducing the cyclic sum symbol $\sum_{cyc}$ implying that the variables a, b, and c, are cycled through, and using a suitable decomposition of the main term, followed by the classical inequality $a^2+b^2 \geq 2ab$, and $\sum_{cyc} a = \sum_{cyc} b = a+b+c$, we get

$$\begin{aligned}
&\frac{a\left(a^2+(1-\ell)b^2\right)}{a^2+kab+b^2}+\frac{b\left(b^2+(1-\ell)c^2\right)}{b^2+kbc+c^2}+\frac{c\left(c^2+(1-\ell)a^2\right)}{c^2+kca+a^2} \\
&= \sum_{cyc} \frac{a\left(a^2+(1-\ell)b^2\right)}{a^2+kab+b^2} = \sum_{cyc}\left(a-\frac{ka^2b+\ell ab^2}{a^2+b^2+kab}\right) \\
&\geq \sum_{cyc}\left(a-\frac{ka^2b+\ell ab^2}{2ab+kab}\right) = \sum_{cyc}\left(a-\frac{ka+\ell b}{2+k}\right) \\
&= \sum_{cyc} a - \frac{k}{2+k}\sum_{cyc} a - \frac{\ell}{2+k}\sum_{cyc} b \\
&= \left(1-\frac{k+\ell}{2+k}\right)\sum_{cyc} a = \frac{2-\ell}{2+k}\sum_{cyc} a = \frac{2-\ell}{2+k}(a+b+c).
\end{aligned}$$

This ends the proof. $\square$

If we take $\ell = 0$, then Theorem 2.1 yields

$$\frac{a\left(a^2+b^2\right)}{a^2+kab+b^2}+\frac{b\left(b^2+c^2\right)}{b^2+kbc+c^2}+\frac{c\left(c^2+a^2\right)}{c^2+kca+a^2} \geq \frac{2}{2+k}(a+b+c).$$

If we take $\ell = 1$, then Theorem 2.1 gives

$$\frac{a^3}{a^2+kab+b^2}+\frac{b^3}{b^2+kbc+c^2}+\frac{c^3}{c^2+kca+a^2} \geq \frac{1}{2+k}(a+b+c).$$

If we take $\ell = 2$, then Theorem 2.1 yields

$$\frac{a\left(a^2-b^2\right)}{a^2+kab+b^2}+\frac{b\left(b^2-c^2\right)}{b^2+kbc+c^2}+\frac{c\left(c^2-a^2\right)}{c^2+kca+a^2} \geq 0.$$

If we take $\ell = 3$, then Theorem 2.1 gives

$$\frac{a\left(a^2-2b^2\right)}{a^2+kab+b^2}+\frac{b\left(b^2-2c^2\right)}{b^2+kbc+c^2}+\frac{c\left(c^2-2a^2\right)}{c^2+kca+a^2} \geq -\frac{1}{2+k}(a+b+c).$$

The theorem below presents our second family of cyclic inequalities.

Theorem 2.2. *For any* $a, b, c > 0$ *and* $k, \ell \geq 0$*, we have*

$$\begin{aligned} &\frac{a^3+a^2b+ab^2(k-\ell+1)+b^3}{a^2+kab+b^2}+\frac{b^3+b^2c+bc^2(k-\ell+1)+c^3}{b^2+kbc+c^2} \\ &+\frac{c^3+c^2a+ca^2(k-\ell+1)+a^3}{c^2+kca+a^2} \geq \frac{4+k-\ell}{2+k}(a+b+c). \end{aligned}$$

Proof. Introducing the cyclic sum symbol $\sum_{cyc}$ and using a suitable decomposition of the main term, followed by the classical inequality $a^2+b^2 \geq 2ab$, and $\sum_{cyc} a = \sum_{cyc} b = a+b+c$, we get

$$\begin{aligned} &\frac{a^3+a^2b+ab^2(k-\ell+1)+b^3}{a^2+kab+b^2}+\frac{b^3+b^2c+bc^2(k-\ell+1)+c^3}{b^2+kbc+c^2} \\ &+\frac{c^3+c^2a+ca^2(k-\ell+1)+a^3}{c^2+kca+a^2} \\ &=\sum_{cyc}\frac{a^3+a^2b+ab^2(k-\ell+1)+b^3}{a^2+kab+b^2}=\sum_{cyc}\left(a+b-\frac{ka^2b+\ell ab^2}{a^2+b^2+kab}\right) \\ &\geq \sum_{cyc}\left(a+b-\frac{ka^2b+\ell ab^2}{2ab+kab}\right)=\sum_{cyc}\left(a+b-\frac{ka+\ell b}{2+k}\right) \end{aligned}$$

$$= \sum_{cyc} a + \sum_{cyc} b - \frac{k}{2+k}\sum_{cyc} a - \frac{\ell}{2+k}\sum_{cyc} b$$
$$= \left(2 - \frac{k+\ell}{2+k}\right)\sum_{cyc} a = \frac{4+k-\ell}{2+k}\sum_{cyc} a = \frac{4+k-\ell}{2+k}(a+b+c).$$

This concludes the proof. □

If we take $k = \ell$, then Theorem 2.2 yields

$$\frac{a^3+a^2b+ab^2+b^3}{a^2+kab+b^2} + \frac{b^3+b^2c+bc^2+c^3}{b^2+kbc+c^2} + \frac{c^3+c^2a+ca^2+a^3}{c^2+kca+a^2} \geq \frac{4}{2+k}(a+b+c).$$

If we take $k = 3$ and $\ell = 1$, then Theorem 2.2 gives

$$\frac{a^3+a^2b+3ab^2+b^3}{a^2+3ab+b^2} + \frac{b^3+b^2c+3bc^2+c^3}{b^2+3bc+c^2} + \frac{c^3+c^2a+3ca^2+a^3}{c^2+3ca+a^2} \geq \frac{6}{5}(a+b+c).$$

Remark 2.1. *The equalities in Theorems 2.1 and 2.2, along with their associated examples, are achieved when $a = b = c$.*

Our third family of cyclic inequalities is stated in the theorem below.

Theorem 2.3. *For any $a, b, c > 0$ and $k, \ell \geq 0$, we have*

$$\frac{a\left(2a^2+ab(2-\ell)+b^2(1-\ell)\right)}{(2+k)a(a+b)+b^2} + \frac{b\left(2b^2+bc(2-\ell)+c^2(1-\ell)\right)}{(2+k)b(b+c)+c^2} + \frac{c\left(2c^2+ca(2-\ell)+a^2(1-\ell)\right)}{(2+k)c(c+a)+a^2} \geq \frac{2-\ell}{2+k}(a+b+c).$$

Proof. Introducing the cyclic sum symbol $\sum_{cyc}$ and using a suitable decomposition of the main term, followed by the inequality $a/(a+b) + (a+b)/a \geq 2$, i.e., $1/z + z \geq 2$ for $z \geq 0$, taken at $z = (a+b)/a$, and $\sum_{cyc} a = \sum_{cyc} b = a+b+c$, we get

$$\begin{aligned}
&\frac{a\left(2a^2+ab(2-\ell)+b^2(1-\ell)\right)}{(2+k)a(a+b)+b^2}+\frac{b\left(2b^2+bc(2-\ell)+c^2(1-\ell)\right)}{(2+k)b(b+c)+c^2}\\
&+\frac{c\left(2c^2+ca(2-\ell)+a^2(1-\ell)\right)}{(2+k)c(c+a)+a^2}\\
&=\sum_{cyc}\frac{a\left(2a^2+ab(2-\ell)+b^2(1-\ell)\right)}{(2+k)a(a+b)+b^2}\\
&=\sum_{cyc}\left(a-\frac{ka^2b+\ell ab^2}{ab\left(a/(a+b)+(a+b)/a\right)+kab}\right)\\
&\geq\sum_{cyc}\left(a-\frac{ka^2b+\ell ab^2}{2ab+kab}\right)=\sum_{cyc}\left(a-\frac{ka+\ell b}{2+k}\right)\\
&=\sum_{cyc}a-\frac{k}{2+k}\sum_{cyc}a-\frac{\ell}{2+k}\sum_{cyc}b\\
&=\left(1-\frac{k+\ell}{2+k}\right)\sum_{cyc}a=\frac{2-\ell}{2+k}\sum_{cyc}a=\frac{2-\ell}{2+k}(a+b+c).
\end{aligned}$$

This completes the proof. □

If we take $\ell=0$, then Theorem 2.3 yields

$$\begin{aligned}
&\frac{a\left(2a^2+2ab+b^2\right)}{(2+k)a(a+b)+b^2}+\frac{b\left(2b^2+2bc+c^2\right)}{(2+k)b(b+c)+c^2}+\frac{c\left(2c^2+2ca+a^2\right)}{(2+k)c(c+a)+a^2}\\
&\geq\frac{2}{2+k}(a+b+c).
\end{aligned}$$

If we take $\ell=1$, then Theorem 2.3 gives

$$\begin{aligned}
&\frac{a^2\left(2a+b\right)}{(2+k)a(a+b)+b^2}+\frac{b^2\left(2b+c\right)}{(2+k)b(b+c)+c^2}+\frac{c^2\left(2c+a\right)}{(2+k)c(c+a)+a^2}\\
&\geq\frac{1}{2+k}(a+b+c).
\end{aligned}$$

If we take $\ell=2$, then Theorem 2.3 yields

$$\frac{a\left(2a^2-b^2\right)}{(2+k)a(a+b)+b^2}+\frac{b\left(2b^2-c^2\right)}{(2+k)b(b+c)+c^2}+\frac{c\left(2c^2-a^2\right)}{(2+k)c(c+a)+a^2}\geq 0.$$

The theorem below presents our fourth family of cyclic inequalities.

Theorem 2.4. *For any $a, b, c > 0$ and $k, \ell \geq 0$, we have*

$$\frac{a^3 + 2a^2b + ab^2(2-\ell) - b^3\ell}{a^2 + (2+k)b(a+b)} + \frac{b^3 + 2b^2c + bc^2(2-\ell) - c^3\ell}{b^2 + (2+k)c(b+c)} + \frac{c^3 + 2c^2a + ca^2(2-\ell) - a^3\ell}{c^2 + (2+k)a(c+a)} \geq \frac{2-\ell}{2+k}(a+b+c).$$

Proof. Introducing the cyclic sum symbol $\sum_{cyc}$ and using a suitable decomposition of the main term, followed by the inequality $b/(a+b) + (a+b)/b \geq 2$, and $\sum_{cyc} a = \sum_{cyc} b = a + b + c$, we get

$$\begin{aligned}
&\frac{a^3 + 2a^2b + ab^2(2-\ell) - b^3\ell}{a^2 + (2+k)b(a+b)} + \frac{b^3 + 2b^2c + bc^2(2-\ell) - c^3\ell}{b^2 + (2+k)c(b+c)} \\
&+ \frac{c^3 + 2c^2a + ca^2(2-\ell) - a^3\ell}{c^2 + (2+k)a(c+a)} \\
&= \sum_{cyc} \frac{a^3 + 2a^2b + ab^2(2-\ell) - b^3\ell}{a^2 + (2+k)b(a+b)} \\
&= \sum_{cyc} \left(a - \frac{ka^2b + \ell ab^2}{ab\left(b/(a+b) + (a+b)/b\right) + kab} \right) \\
&\geq \sum_{cyc} \left(a - \frac{ka^2b + \ell ab^2}{2ab + kab} \right) = \sum_{cyc} \left(a - \frac{ka + \ell b}{2+k} \right) \\
&= \sum_{cyc} a - \frac{k}{2+k} \sum_{cyc} a - \frac{\ell}{2+k} \sum_{cyc} b \\
&= \left(1 - \frac{k+\ell}{2+k} \right) \sum_{cyc} a = \frac{2-\ell}{2+k} \sum_{cyc} a = \frac{2-\ell}{2+k}(a+b+c).
\end{aligned}$$

This concludes the proof. □

If we take $\ell = 0$, then Theorem 2.4 gives

$$\frac{a\left(a^2 + 2ab + 2b^2\right)}{a^2 + (2+k)b(a+b)} + \frac{b\left(b^2 + 2bc + 2c^2\right)}{b^2 + (2+k)c(b+c)} + \frac{c\left(c^2 + 2ca + 2a^2\right)}{c^2 + (2+k)a(c+a)} \geq \frac{2}{2+k}(a+b+c).$$

If we take $\ell = 1$, then Theorem 2.4 yields

$$\frac{a^3+2a^2b+ab^2-b^3}{a^2+(2+k)b(a+b)}+\frac{b^3+2b^2c+bc^2-c^3}{b^2+(2+k)c(b+c)}$$
$$+\frac{c^3+2c^2a+ca^2-a^3}{c^2+(2+k)a(c+a)}\geq\frac{1}{2+k}(a+b+c).$$

If we take $\ell=2$, then Theorem 2.4 gives

$$\frac{a^3+2a^2b-2b^3}{a^2+(2+k)b(a+b)}+\frac{b^3+2b^2c-2c^3}{b^2+(2+k)c(b+c)}+\frac{c^3+2c^2a-2a^3}{c^2+(2+k)a(c+a)}\geq 0.$$

Our fifth family of cyclic inequalities is stated in the theorem below.

Theorem 2.5. *For any $a,b,c>0$ and $k,\ell\geq 0$, we have*

$$\frac{(a+b)\left(2a^2+ab(k-\ell+2)+b^2\right)}{(2+k)a(a+b)+b^2}+\frac{(b+c)\left(2b^2+bc(k-\ell+2)+c^2\right)}{(2+k)b(b+c)+c^2}$$
$$+\frac{(c+a)\left(2c^2+ca(k-\ell+2)+a^2\right)}{(2+k)c(c+a)+a^2}\geq\frac{4+k-\ell}{2+k}(a+b+c).$$

Proof. Introducing the cyclic sum symbol $\sum_{cyc}$ and using a suitable decomposition of the main term, followed by the inequality $a/(a+b)+(a+b)/a\geq 2$, and $\sum_{cyc}a=\sum_{cyc}b=a+b+c$, we get

$$\frac{(a+b)\left(2a^2+ab(k-\ell+2)+b^2\right)}{(2+k)a(a+b)+b^2}+\frac{(b+c)\left(2b^2+bc(k-\ell+2)+c^2\right)}{(2+k)b(b+c)+c^2}$$
$$+\frac{(c+a)\left(2c^2+ca(k-\ell+2)+a^2\right)}{(2+k)c(c+a)+a^2}$$
$$=\sum_{cyc}\frac{(a+b)\left(2a^2+ab(k-\ell+2)+b^2\right)}{(2+k)a(a+b)+b^2}$$
$$=\sum_{cyc}\left(a+b-\frac{ka^2b+\ell ab^2}{ab\left(a/(a+b)+(a+b)/a\right)+kab}\right)$$
$$\geq\sum_{cyc}\left(a+b-\frac{ka^2b+\ell ab^2}{2ab+kab}\right)=\sum_{cyc}\left(a+b-\frac{ka+\ell b}{2+k}\right)$$
$$=\sum_{cyc}a+\sum_{cyc}b-\frac{k}{2+k}\sum_{cyc}a-\frac{\ell}{2+k}\sum_{cyc}b$$
$$=\left(2-\frac{k+\ell}{2+k}\right)\sum_{cyc}a=\frac{4+k-\ell}{2+k}\sum_{cyc}a=\frac{4+k-\ell}{2+k}(a+b+c).$$

This completes the proof. □

If we take $k=\ell$, then Theorem 2.5 yields

$$\frac{(a+b)\left(2a^2+2ab+b^2\right)}{(2+k)a(a+b)+b^2}+\frac{(b+c)\left(2b^2+2bc+c^2\right)}{(2+k)b(b+c)+c^2}$$
$$+\frac{(c+a)\left(2c^2+2ca+a^2\right)}{(2+k)c(c+a)+a^2}\geq\frac{4}{2+k}(a+b+c).$$

If we take $k=3$ and $\ell=1$, then Theorem 2.5 gives

$$\frac{(a+b)\left(2a^2+4ab+b^2\right)}{5a(a+b)+b^2}+\frac{(b+c)\left(2b^2+4bc+c^2\right)}{5b(b+c)+c^2}$$
$$+\frac{(c+a)\left(2c^2+4ca+a^2\right)}{5c(c+a)+a^2}\geq\frac{6}{5}(a+b+c).$$

The theorem below states our sixth and final family of cyclic inequalities.

Theorem 2.6. *For any $a,b,c>0$ and $k,\ell\geq 0$, we have*

$$\frac{(a+b)\left(a^2+2ab+b^2(k-\ell+2)\right)}{a^2+(2+k)b(a+b)}+\frac{(b+c)\left(b^2+2bc+c^2(k-\ell+2)\right)}{b^2+(2+k)c(b+c)}$$
$$+\frac{(c+a)\left(c^2+2ca+a^2(k-\ell+2)\right)}{c^2+(2+k)a(c+a)}\geq\frac{4+k-\ell}{2+k}(a+b+c).$$

Proof. Introducing the cyclic sum symbol $\sum_{cyc}$ and using a suitable decomposition of the main term, followed by the inequality $b/(a+b)+(a+b)/b\geq 2$, and $\sum_{cyc}a=\sum_{cyc}b=a+b+c$, we get

$$\frac{(a+b)\left(a^2+2ab+b^2(k-\ell+2)\right)}{a^2+(2+k)b(a+b)}+\frac{(b+c)\left(b^2+2bc+c^2(k-\ell+2)\right)}{b^2+(2+k)c(b+c)}$$
$$+\frac{(c+a)\left(c^2+2ca+a^2(k-\ell+2)\right)}{c^2+(2+k)a(c+a)}$$
$$=\sum_{cyc}\frac{(a+b)\left(a^2+2ab+b^2(k-\ell+2)\right)}{a^2+(2+k)b(a+b)}$$
$$=\sum_{cyc}\left(a+b-\frac{ka^2b+\ell ab^2}{ab\left(b/(a+b)+(a+b)/b\right)+kab}\right)$$
$$\geq\sum_{cyc}\left(a+b-\frac{ka^2b+\ell ab^2}{2ab+kab}\right)=\sum_{cyc}\left(a+b-\frac{ka+\ell b}{2+k}\right)$$

$$
\begin{aligned}
&= \sum_{cyc} a + \sum_{cyc} b - \frac{k}{2+k} \sum_{cyc} a - \frac{\ell}{2+k} \sum_{cyc} b \\
&= \left(2 - \frac{k+\ell}{2+k}\right) \sum_{cyc} a = \frac{4+k-\ell}{2+k} \sum_{cyc} a = \frac{4+k-\ell}{2+k}(a+b+c).
\end{aligned}
$$

This ends the proof. □

If we take $k = \ell$, then Theorem 2.6 yields

$$
\begin{aligned}
&\frac{(a+b)\left(a^2+2ab+2b^2\right)}{a^2+(2+k)b(a+b)} + \frac{(b+c)\left(b^2+2bc+2c^2\right)}{b^2+(2+k)c(b+c)} \\
&+ \frac{(c+a)\left(c^2+2ca+2a^2\right)}{c^2+(2+k)a(c+a)} \geq \frac{4}{2+k}(a+b+c).
\end{aligned}
$$

If we take $k = 3$ and $\ell = 1$, then Theorem 2.6 gives

$$
\begin{aligned}
&\frac{(a+b)\left(a^2+2ab+4b^2\right)}{a^2+5b(a+b)} + \frac{(b+c)\left(b^2+2bc+4c^2\right)}{b^2+5c(b+c)} \\
&+ \frac{(c+a)\left(c^2+2ca+4a^2\right)}{c^2+5a(c+a)} \geq \frac{6}{5}(a+b+c).
\end{aligned}
$$

We hope these findings will inspire the exploration of broader classes of cyclic inequalities featuring more sophisticated structures.

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