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SEVERAL NEW CONVEX INTEGRAL INEQUALITIES

Christophe Chesneau

ABSTRACT. This paper presents a new convexity result involving the primitive of a function, from which we derive several new integral inequalities.

1. INTRODUCTION

Convex and concave functions are fundamental to mathematical analysis. For rigor and completeness, we establish their formal definitions below. Let $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $b > a$.

Convex function: A function $f : [a, b] \to \mathbb{R}$ is said to be convex if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y). \tag{1.1}$$

Concave function: A function $f : [a, b] \to \mathbb{R}$ is said to be concave if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y).$$

A primary application of convexity and concavity lies in the formulation of fundamental integral inequalities, such as the celebrated Jensen and Hermite-Hadamard integral inequalities. For comprehensive surveys and recent results

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in this domain, we refer the reader to an extensive body of literature (see, e.g., [1–17]).

In this paper, we establish a new and simple convex result involving the primitive of a function. By combining this framework with the classical Jensen and Hermite-Hadamard integral inequalities, and the weighted Jensen inequality, we derive new integral inequalities. We also briefly discuss the analogous concave case and provide comprehensive proofs for all main results.

2. Results

2.1. A convex result. A new convex result involving the primitive of a function is presented in the theorem below.

Theorem 2.1. *Let $f : (0, +\infty) \to \mathbb{R}$ be a convex function, and*

$$g(x) = \frac{1}{x} \int_0^x f(t)dt, \quad x > 0,$$

provided the integral term exists. Then g is convex.

Proof. For any $\lambda \in [0, 1]$, and $x, y > 0$, using the change of variables $t = (\lambda x + (1 - \lambda)y)u$, the convexity of f characterized by the inequality in Equation (1.1), followed by the two changes of variables $v = xu$ and $w = yu$, we have

$$\begin{aligned}
g(\lambda x + (1 - \lambda)y) &= \frac{1}{\lambda x + (1 - \lambda)y} \int_0^{\lambda x + (1-\lambda)y} f(t)dt \\
&= \int_0^1 f((\lambda x + (1 - \lambda)y)u)du \\
&= \int_0^1 f(\lambda xu + (1 - \lambda)yu)du \\
&\leq \int_0^1 (\lambda f(xu) + (1 - \lambda)f(yu))\, du \\
&= \lambda \int_0^1 f(xu)du + (1 - \lambda) \int_0^1 f(yu)du \\
&= \lambda \cdot \frac{1}{x} \int_0^x f(v)dv + (1 - \lambda) \cdot \frac{1}{y} \int_0^y f(w)dw \\
&= \lambda g(x) + (1 - \lambda)g(y).
\end{aligned}$$

Therefore, the inequality in Equation (1.1) is satisfied by g, implying that g is convex. This concludes the proof. $\square$

Note that

- if f is concave instead of convex, the sole inequality in the proof of Theorem 2.1 is reversed, implying that g is also concave,
- if f is twice differentiable and convex, i.e., for any $x > 0$, we have $f''(x) \geq 0$, then two differentiations completed by two integrations by parts give
$$g''(x) = \frac{1}{x^3} \int_0^x t^2 f''(t)dt \geq 0,$$
recovering the conclusion that g is convex,
- if f is differentiable and increasing, then the primitive function $F(x) = xg(x) = \int_0^x f(t)dt$ is convex, which is a known result that comes immediately from two differentiations, i.e., $F''(x) = f'(x) \geq 0$.

2.2. **Integral inequalities.** Our first integral inequality result is given in the theorem below. It is an application of Theorem 2.1 and the Jensen integral inequality.

Theorem 2.2. *Let $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $b > a$, $f : (0, +\infty) \to \mathbb{R}$ be a convex function and $h, w : [a, b] \to [0, +\infty)$ be two functions with $\int_a^b w(x)dx = 1$. Then we have*
$$\int_0^{\int_a^b h(x)w(x)dx} f(t)dt \leq \left(\int_a^b h(x)w(x)dx\right) \int_a^b \frac{1}{h(x)} \left(\int_0^{h(x)} f(t)dt\right) w(x)dx,$$
provided all the integral terms exist.

Proof. The inequality holds trivially if $\int_a^b h(x)w(x)dx = 0$. Suppose instead that $\int_a^b h(x)w(x)dx > 0$. Let us set
$$g(x) = \frac{1}{x} \int_0^x f(t)dt, \quad x > 0.$$
Theorem 2.1 ensures that g is convex. Therefore, the Jensen integral inequality applied to g gives
$$g\left(\int_a^b h(x)w(x)dx\right) \leq \int_a^b g(h(x))w(x)dx,$$

that is

$$\frac{1}{\int_a^b h(x)w(x)dx}\int_0^{\int_a^b h(x)w(x)dx} f(t)dt \leq \int_a^b \frac{1}{h(x)}\left(\int_0^{h(x)} f(t)dt\right) w(x)dx.$$

Since $\int_a^b h(x)w(x)dx > 0$, this is equivalent to

$$\int_0^{\int_a^b h(x)w(x)dx} f(t)dt \leq \left(\int_a^b h(x)w(x)dx\right)\int_a^b \frac{1}{h(x)}\left(\int_0^{h(x)} f(t)dt\right) w(x)dx.$$

This ends the proof. □

Note that

- if f is concave instead of convex, then the inequality is reversed,
- if $h = w$, so that $\int_a^b h(x)dx = 1$, then we have

$$\int_0^{\int_a^b h^2(x)dx} f(t)dt \leq \left(\int_a^b h^2(x)dx\right)\int_a^b \left(\int_0^{h(x)} f(t)dt\right) dx.$$

New integral inequalities are given in the theorem below. There are applications of Theorem 2.1 and the Hermite-Hadamard integral (double) inequality.

Theorem 2.3. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f : (0, b) \to \mathbb{R}$ be a convex integrable function. Then we have*

$$\frac{2}{a+b}\int_0^{(a+b)/2} f(x)dx \leq \frac{1}{b-a}\int_a^b \frac{1}{x}\left(\int_0^x f(t)dt\right) dx$$
$$\leq \frac{1}{2}\left(\frac{1}{a}\int_0^a f(t)dt + \frac{1}{b}\int_0^b f(t)dt\right).$$

Proof. Let us set

$$g(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0.$$

Theorem 2.1 ensures that g is convex. Therefore, the Hermite-Hadamard integral inequality applied to g gives

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(x)dx \leq \frac{1}{2}\left(g(a) + g(b)\right),$$

that is

$$\frac{2}{a+b}\int_0^{(a+b)/2} f(x)dx \le \frac{1}{b-a}\int_a^b \frac{1}{x}\left(\int_0^x f(t)dt\right)dx$$
$$\le \frac{1}{2}\left(\frac{1}{a}\int_0^a f(t)dt + \frac{1}{b}\int_0^b f(t)dt\right).$$

This ends the proof. □

Note that if f is concave instead of convex, then the inequalities are reversed.

A new integral inequality is presented in the theorem below. It is an application of Theorem 2.1 and the weighted Jensen inequality.

Theorem 2.4. *Let $a, b, c > 0$ and $f : (0, +\infty) \to \mathbb{R}$ be a convex function. Then we have*

$$\frac{a}{b+c}\int_0^{b+c} f(t)dt + \frac{b}{c+a}\int_0^{c+a} f(t)dt + \frac{c}{a+b}\int_0^{a+b} f(t)dt$$
$$\ge \frac{(a+b+c)^2}{2(ab+bc+ca)}\int_0^{2(ab+bc+ca)/(a+b+c)} f(t)dt,$$

provided the integral terms exist.

Proof. Let us set

$$g(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0.$$

Theorem 2.1 ensures that g is convex. Therefore, the weighted Jensen inequality applied to g gives

$$ag(b+c) + bg(c+a) + cg(a+b)$$
$$= (a+b+c)\left(\frac{a}{a+b+c}g(b+c) + \frac{b}{a+b+c}g(c+a) + \frac{c}{a+b+c}g(a+b)\right)$$
$$\ge (a+b+c)g\left(\frac{a(b+c)+b(c+a)+c(a+b)}{a+b+c}\right)$$
$$= (a+b+c)g\left(\frac{2(ab+bc+ca)}{a+b+c}\right),$$

which is equivalent to

$$\begin{aligned}
&\frac{a}{b+c}\int_0^{b+c} f(t)dt + \frac{b}{c+a}\int_0^{c+a} f(t)dt + \frac{c}{a+b}\int_0^{a+b} f(t)dt \\
&\geq \frac{(a+b+c)^2}{2(ab+bc+ca)}\int_0^{2(ab+bc+ca)/(a+b+c)} f(t)dt.
\end{aligned}$$

This completes the proof. □

Note that

- if f is concave instead of convex, then the inequality is reversed,
- if f is positive, using the known inequality $ab+bc+ca \leq (1/3)(a+b+c)^2$, then we get

$$\begin{aligned}
&\frac{a}{b+c}\int_0^{b+c} f(t)dt + \frac{b}{c+a}\int_0^{c+a} f(t)dt + \frac{c}{a+b}\int_0^{a+b} f(t)dt \\
&\geq \frac{(a+b+c)^2}{2(ab+bc+ca)}\int_0^{2(ab+bc+ca)/(a+b+c)} f(t)dt \\
&\geq \frac{3}{2}\int_0^{2(ab+bc+ca)/(a+b+c)} f(t)dt,
\end{aligned}$$

  which may be viewed as a modification of the famous Nesbitt inequality.

As an example, if we take $f(x) = e^x$, which is convex, Theorem 2.4 yields

$$\begin{aligned}
&\frac{a}{b+c}(e^{b+c}-1) + \frac{b}{c+a}(e^{c+a}-1) + \frac{c}{a+b}(e^{a+b}-1) \\
&\geq \frac{(a+b+c)^2}{2(ab+bc+ca)}\left(e^{2(ab+bc+ca)/(a+b+c)}-1\right).
\end{aligned}$$

These integral inequalities highlight the potential of Theorem 2.1 to expand traditional frameworks in convex analysis.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CAEN-NORMANDIE
UFR DES SCIENCES - CAMPUS 2, CAEN
FRANCE.
Email address: christophe.chesneau@gmail.com