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A NOTE ON A GENERAL CONVEX CYCLIC INEQUALITY

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ABSTRACT. We introduce a general family of cyclic inequalities parameterized by a convex function. Furthermore, we present several applications and establish a broader extension.

1. INTRODUCTION

Cyclic inequalities represent a foundational class of mathematical problems, often serving as a bridge between elementary algebraic manipulation and advanced analytical techniques. A key result of this study is the Nesbitt inequality introduced in [6], wich states that, for any $a, b, c > 0$,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

The equality holds for $a = b = c$. This inequality exemplifies the mathematical elegance inherent in cyclic structures, where the expression remains invariant under cyclic permutations of the variables. While the Nesbitt inequality provides a classic benchmark, modern investigations often seek to generalize such results. See [1–7] and the references therein.

In this paper, we present a general cyclic inequality that incorporates an adjustable convex function, f. The idea of integrating such functions into cyclic

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structures is to establish a unified framework for generating sophisticated analytic results from fundamental functional principles. We contribute to this line of research by exploring a specific cyclic structure of interest. To further motivate our approach, an elegant three-variable analog of the Nesbitt inequality as a direct consequence of our main result applied to a specific convex function is presented below. For any $a, b, c > 0$, we have

$$\frac{a}{b+2a+3c} + \frac{b}{c+2b+3a} + \frac{c}{a+2c+3b} \geq \frac{1}{2}.$$

The equality holds for $a = b = c$. We highlight the cyclic weighting by the coefficients 1, 2, and 3 across the variables a, b, and c in the denominators, which endows this inequality with a particularly attractive structure. The central tool in our proofs is a suitable application of the Jensen inequality.

The remainder of the paper is organized as follows: The main result is developed in Section 2, followed by applications in Section 3. Finally, a further extension of this result, also derived from the convexity approach, is presented in Section 4.

2. Main result

We present our main result in the theorem below, emphasizing the central role of the convex function f in the formulation.

Theorem 2.1. *Let $f : (0, \infty) \to \mathbb{R}$ be a convex function. Then, for any $a, b, c > 0$, the following cyclic inequality holds:*

$$af(b+2a+3c) + bf(c+2b+3a) + cf(a+2c+3b) \geq (a+b+c)f\left(2(a+b+c)\right).$$

The equality holds for $a = b = c$.

Proof. By adjusting the weights and applying the Jensen inequality to the convex function f, we obtain

$$\begin{aligned} &af(b+2a+3c) + bf(c+2b+3a) + cf(a+2c+3b) \\ &= (a+b+c) \\ &\times \left(\frac{a}{a+b+c} f(b+2a+3c) + \frac{b}{a+b+c} f(c+2b+3a) + \frac{c}{a+b+c} f(a+2c+3b) \right) \end{aligned}$$

$$
\begin{aligned}
&\overset{Jensen's\ inequality}{\geq} (a+b+c)f\left(\frac{a(b+2a+3c)}{a+b+c}+\frac{b(c+2b+3a)}{a+b+c}+\frac{c(a+2c+3b)}{a+b+c}\right) \\
&= (a+b+c)f\left(\frac{a(b+2a+3c)+b(c+2b+3a)+c(a+2c+3b)}{a+b+c}\right) \\
&= (a+b+c)f\left(\frac{2(a^2+b^2+c^2)+4(ab+bc+ca)}{a+b+c}\right) \\
&= (a+b+c)f\left(\frac{2(a+b+c)^2}{a+b+c}\right) \\
&= (a+b+c)f\left(2(a+b+c)\right).
\end{aligned}
$$

Clearly, the equality holds for $a = b = c$, with both sides of the inequality reducing to $3af(6a)$. This concludes the proof. □

Equivalently, our result can be restated using cyclic sum notation as

$$\sum_{cyc} af(b+2a+3c) \geq (a+b+c)f\left(2(a+b+c)\right),$$

where the cyclic permutation is taken as $a \to b \to c$.

Note that the Jensen inequality is the sole inequality used in the proof. Consequently, f need only be convex; the absence of a monotonicity assumption makes this result particularly interesting from a theoretical perspective.

Furthermore, if the function f is concave instead of convex, then the inequality is reversed.

We present a generalization of Theorem 2.1 incorporating two parameters, γ and θ. We omit the proof here for the sake of brevity, as it is analogous to the proof of Theorem 2.1.

Theorem 2.2. *Let $f : (0,\infty) \to \mathbb{R}$ be a convex function. Then, for any $a, b, c, \gamma, \theta > 0$ with*

$$\gamma + \theta = 4,$$

the following cyclic inequality holds:

$$
\begin{aligned}
&af(\gamma b+2a+\theta c)+bf(\gamma c+2b+\theta a)+cf(\gamma a+2c+\theta b) \\
&\geq (a+b+c)f\left(2(a+b+c)\right).
\end{aligned}
$$

The equality holds for $a = b = c$.

Equivalently, our result can be restated using cyclic sum notation as

$$\sum_{cyc} af(\gamma b + 2a + \theta c) \geq (a+b+c) f\left(2(a+b+c)\right).$$

The inclusion of the parameters γ and θ provides significant flexibility in the application of this framework. In particular, setting $\gamma = 3$ and $\theta = 1$, we have

$$af(c+2a+3b) + bf(a+2b+3c) + cf(b+2c+3a) \geq (a+b+c) f\left(2(a+b+c)\right),$$

which may be viewed as the twin cyclic inequality of Theorem 2.1.

3. Applications

We illustrate the utility of Theorem 2.1 by applying it to specific choices of the convex function f.

The function $f(x) = x^p$ with $p \in \mathbb{R}\backslash(0,1)$ is convex (for any $x \in (0,\infty)$). It follows from Theorem 2.1 that

$$a(b+2a+3c)^p + b(c+2b+3a)^p + c(a+2c+3b)^p \geq (a+b+c)\left(2(a+b+c)\right)^p$$
$$= 2^p(a+b+c)^{p+1}.$$

In particular,

- setting $p = 2$, we have
$$a(b+2a+3c)^2 + b(c+2b+3a)^2 + c(a+2c+3b)^2 \geq 4(a+b+c)^3,$$
- setting $p = -1/2$, we have
$$\frac{a}{\sqrt{b+2a+3c}} + \frac{b}{\sqrt{c+2b+3a}} + \frac{c}{\sqrt{a+2c+3b}} \geq \frac{1}{\sqrt{2}}\sqrt{a+b+c},$$
- setting $p = -1$, we have
$$\frac{a}{b+2a+3c} + \frac{b}{c+2b+3a} + \frac{c}{a+2c+3b} \geq \frac{1}{2},$$
which can be viewed as a three-variable analogue of the Nesbitt inequality, as outlined in the introduction,
- setting $p = -2$, we have
$$\frac{a}{(b+2a+3c)^2} + \frac{b}{(c+2b+3a)^2} + \frac{c}{(a+2c+3b)^2} \geq \frac{1}{4(a+b+c)}.$$

The function $f(x) = e^{qx}$ with $q \in \mathbb{R}$ is convex. It follows from Theorem 2.1 that

$$ae^{q(b+2a+3c)} + be^{q(c+2b+3a)} + ce^{q(a+2c+3b)} \geq (a+b+c)e^{2q(a+b+c)}.$$

In particular,

- setting $q = 1$, we have

$$ae^{b+2a+3c} + be^{c+2b+3a} + ce^{a+2c+3b} \geq (a+b+c)e^{2(a+b+c)},$$

- setting $q = -1$, we have

$$ae^{-(b+2a+3c)} + be^{-(c+2b+3a)} + ce^{-(a+2c+3b)} \geq (a+b+c)e^{-2(a+b+c)}.$$

The function $f(x) = -\ln(x)$ is convex. It follows from Theorem 2.1 that

$$\begin{aligned}&- a\ln(b+2a+3c) - b\ln(c+2b+3a) - c\ln(a+2c+3b)\\ &\geq -(a+b+c)\ln\left(2(a+b+c)\right),\end{aligned}$$

which is equivalent to

$$(b+2a+3c)^a(c+2b+3a)^b(a+2c+3b)^c \leq 2^{a+b+c}(a+b+c)^{a+b+c}.$$

The function $f(x) = \ln(1+e^{qx})$ with $q \in \mathbb{R}$ is convex. It follows from Theorem 2.1 that

$$\begin{aligned}&a\ln\left(1+e^{q(b+2a+3c)}\right) + b\ln\left(1+e^{q(c+2b+3a)}\right) + c\ln\left(1+e^{q(a+2c+3b)}\right)\\ &\geq (a+b+c)\ln\left(1+e^{2q(a+b+c)}\right),\end{aligned}$$

which is equivalent to

$$\left(1+e^{q(b+2a+3c)}\right)^a\left(1+e^{q(c+2b+3a)}\right)^b\left(1+e^{q(a+2c+3b)}\right)^c \geq \left(1+e^{2q(a+b+c)}\right)^{a+b+c}.$$

In particular,

- setting $q = 1$, we have

$$\left(1+e^{b+2a+3c}\right)^a\left(1+e^{c+2b+3a}\right)^b\left(1+e^{a+2c+3b}\right)^c \geq \left(1+e^{2(a+b+c)}\right)^{a+b+c},$$

- setting $q = -1$, we have

$$\left(1+e^{-(b+2a+3c)}\right)^a\left(1+e^{-(c+2b+3a)}\right)^b\left(1+e^{-(a+2c+3b)}\right)^c \geq \left(1+e^{-2(a+b+c)}\right)^{a+b+c}.$$

As a last and general example, the primitive function $f(x) = \int_0^x g(t)dt$, where $g : (0,\infty) \to \mathbb{R}$ denotes a differentiable increasing function, is convex. It follows from Theorem 2.1 that

$$\begin{aligned}&a\int_0^{b+2a+3c} g(t)dt + b\int_0^{c+2b+3a} g(t)dt + c\int_0^{a+2c+3b} g(t)dt\\ &\geq (a+b+c)\int_0^{2(a+b+c)} g(t)dt.\end{aligned}$$

Abstract cyclic inequalities can also be established for functions that lack elementary primitives. For example, by choosing $g(t) = (2/\sqrt{\pi})e^{t^2}$ and considering the imaginary error function $\mathrm{erfi}(x) = (2/\sqrt{\pi})\int_0^x e^{t^2}dt,\ x \geq 0$, Theorem 2.1 yields

$$a\mathrm{erfi}(b+2a+3c)+b\mathrm{erfi}(c+2b+3a)+c\mathrm{erfi}(a+2c+3b) \geq (a+b+c)\mathrm{erfi}\left(2(a+b+c)\right),$$

which provides a non-trivial inequality for special functions without requiring explicit integration.

4. Extension

An extension of Theorem 2.1 is presented in the theorem below, emphasizing the central role of the convex and increasing function h in the formulation.

Theorem 4.1. *Let $f : (0,\infty) \to \mathbb{R}$ be a convex function, and $h : \mathbb{R} \to \mathbb{R}$ be a convex and increasing function. Then, for any $a, b, c > 0$, the following cyclic inequality holds:*

$$\begin{aligned}&h\left(af(b+2a+3c)\right) + h\left(bf(c+2b+3a)\right) + h\left(cf(a+2c+3b)\right)\\ &\geq 3h\left(\frac{a+b+c}{3}f\left(2(a+b+c)\right)\right).\end{aligned}$$

The equality holds for $a = b = c$.

Proof. By adjusting the weights, applying the Jensen inequality to the convex function h and using Theorem 2.1 combined with the increasing property of h, we obtain

$$\begin{aligned}&h\left(af(b+2a+3c)\right) + h\left(bf(c+2b+3a)\right) + h\left(cf(a+2c+3b)\right)\\ &= 3\left(\frac{1}{3}h\left(af(b+2a+3c)\right) + \frac{1}{3}h\left(bf(c+2b+3a)\right) + \frac{1}{3}h\left(cf(a+2c+3b)\right)\right)\end{aligned}$$

$$\overset{Jensen's\ inequality}{\geq} 3h\left(\frac{1}{3}af(b+2a+3c)+\frac{1}{3}bf(c+2b+3a)+\frac{1}{3}cf(a+2c+3b)\right)$$
$$= 3h\left(\frac{af(b+2a+3c)+bf(c+2b+3a)+cf(a+2c+3b)}{3}\right)$$
$$\overset{Theorem\ 2.1\ and\ increasing\ property}{\geq} 3h\left(\frac{a+b+c}{3}f\left(2(a+b+c)\right)\right).$$

Clearly, the equality holds for $a = b = c$, with both sides of the inequality reducing to $3h\left(af(6a)\right)$. This completes the proof. $\square$

Equivalently, our result can be restated using cyclic sum notation as

$$\sum_{cyc} h\left(af(b+2a+3c)\right) \geq 3h\left(\frac{a+b+c}{3}f\left(2(a+b+c)\right)\right).$$

Some examples of application of Theorem 4.1 considering a positive function f are now presented.

As the first example, the function $h(x) = x^p$ with $p \geq 1$ is convex and increasing. It follows from Theorem 4.1 that

$$\begin{aligned}&(af(b+2a+3c))^p+(bf(c+2b+3a))^p+(cf(a+2c+3b))^p\\&\geq 3\left(\frac{a+b+c}{3}f\left(2(a+b+c)\right)\right)^p,\end{aligned}$$

so that

$$\begin{aligned}&a^pf(b+2a+3c)^p+b^pf(c+2b+3a)^p+c^pf(a+2c+3b)^p\\&\geq \frac{(a+b+c)^p}{3^{p-1}}f\left(2(a+b+c)\right)^p.\end{aligned}$$

This provides a one-parameter extension of Theorem 2.1. Setting $f(x) = 1/x$, we also obtain the following elegant cyclic inequality:

$$\frac{a^p}{(b+2a+3c)^p}+\frac{b^p}{(c+2b+3a)^p}+\frac{c^p}{(a+2c+3b)^p} \geq \frac{1}{3^{p-1}2^p}.$$

In particular, setting $p = 2$, we get

$$\frac{a^2}{(b+2a+3c)^2}+\frac{b^2}{(c+2b+3a)^2}+\frac{c^2}{(a+2c+3b)^2} \geq \frac{1}{12}.$$

As another example, the function $h(x) = e^x$ is convex and increasing. It follows from Theorem 4.1 that

$$e^{af(b+2a+3c)} + e^{bf(c+2b+3a)} + e^{cf(a+2c+3b)} \geq 3e^{((a+b+c)/3)f(2(a+b+c))}.$$

Setting $f(x) = 1/x$, we get

$$e^{a/(b+2a+3c)} + e^{b/(c+2b+3a)} + e^{c/(a+2c+3b)} \geq 3e^{1/6}.$$

As a final example, the function $h(x) = x\ln(1+x)$ is convex and increasing. It follows from Theorem 4.1 that

$$\begin{aligned}
&af(b+2a+3c)\ln(1+af(b+2a+3c)) + bf(c+2b+3a)\ln(1+bf(c+2b+3a)) \\
&+ cf(a+2c+3b)\ln(1+cf(a+2c+3b)) \\
&\geq 3\left(\frac{a+b+c}{3}f(2(a+b+c))\right)\ln\left(1+\frac{a+b+c}{3}f(2(a+b+c))\right) \\
&= (a+b+c)f(2(a+b+c))\ln\left(1+\frac{a+b+c}{3}f(2(a+b+c))\right),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&(1+af(b+2a+3c))^{af(b+2a+3c)}(1+bf(c+2b+3a))^{bf(c+2b+3a)} \\
&\times (1+cf(a+2c+3b))^{cf(a+2c+3b)} \\
&\geq \left(1+\frac{a+b+c}{3}f(2(a+b+c))\right)^{(a+b+c)f(2(a+b+c))}.
\end{aligned}$$

Setting $f(x) = 1/x$, we get the elegant product inequality

$$\begin{aligned}
&\left(1+\frac{a}{b+2a+3c}\right)^{a/(b+2a+3c)}\left(1+\frac{b}{c+2b+3a}\right)^{b/(c+2b+3a)} \\
&\times\left(1+\frac{c}{a+2c+3b}\right)^{c/(a+2c+3b)} \geq \sqrt{\frac{7}{6}}.
\end{aligned}$$

The inclusion of the function h facilitates a much wider variety of such examples.

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